The meaning of Distances in Spectral Analysis

46$^{\mathrm{th}}$ IEEE Conference on Decision & Control

Plenary presentation

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Electrical & Computer Engineering
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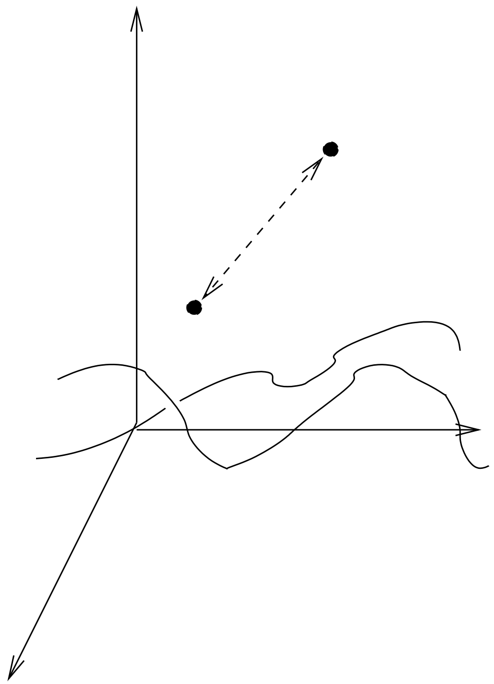
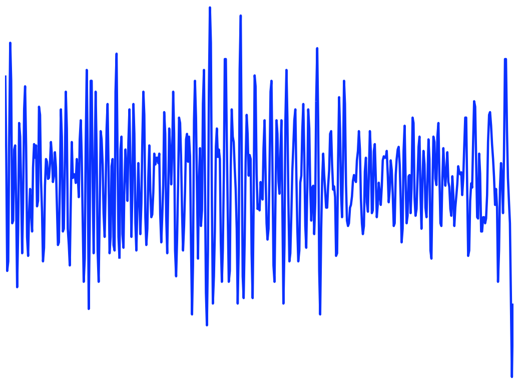
Meaning of distances

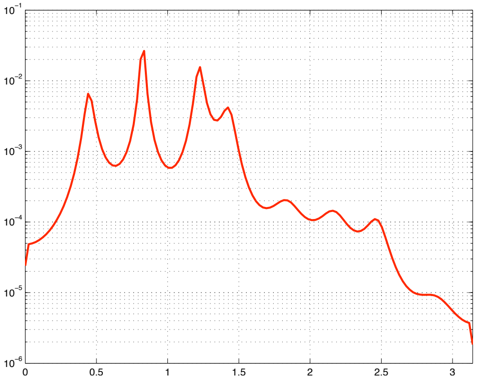


- maximal separation (L_∞)
- energy-like content (L_2)
- integral of flow-rate (L_1)

Power spectra

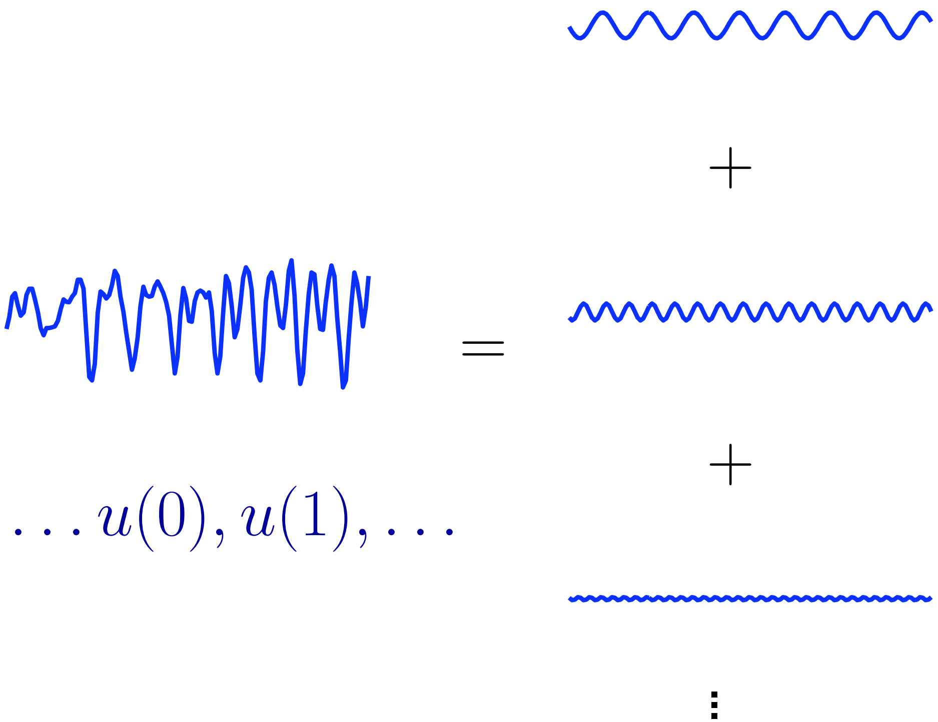


$\Rightarrow$

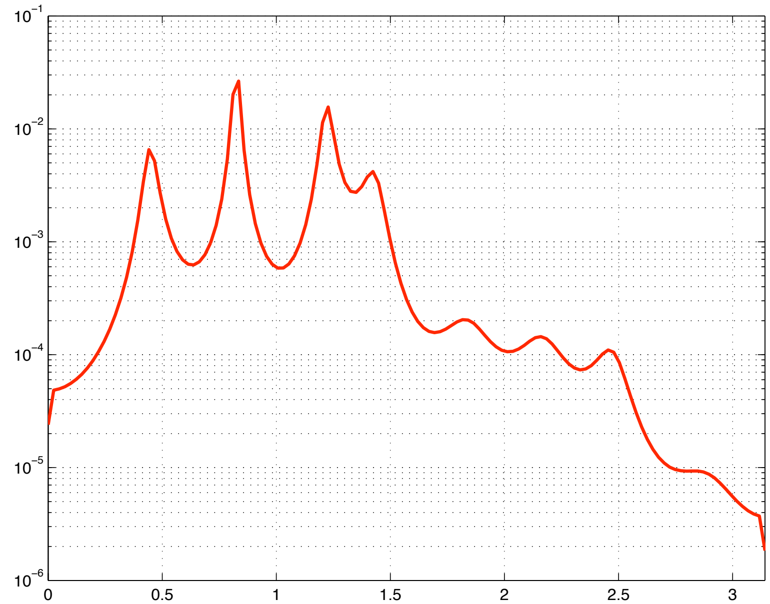


Periodogram, Blackman-Tukey, Levinson, Durbin, Burg, . . .

Spectral analysis



$\ldots u(0), u(1), \ldots$

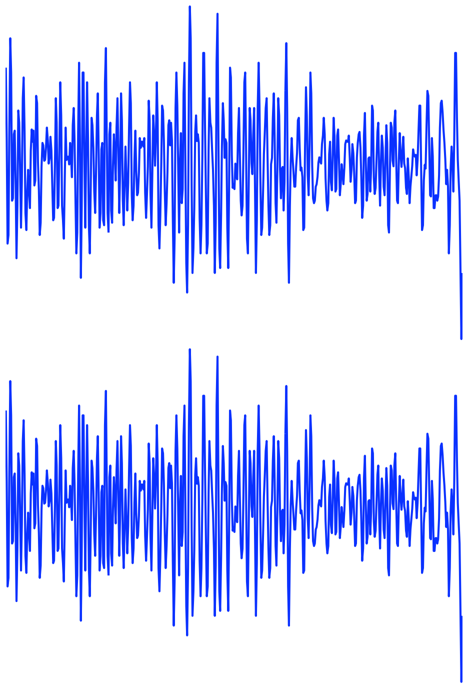


$$u(k) = \int e^{jk\theta} dX(\theta)$$

$$E\{u(k)u(k+\ell)\} = \int e^{j\ell\theta} f(\theta) d\theta$$

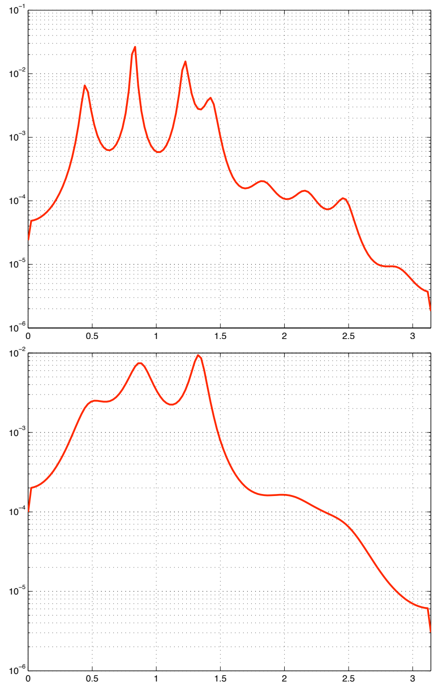
Signals vs. power densities

time-signals



$(u_1 - u_2)$ "error signal"

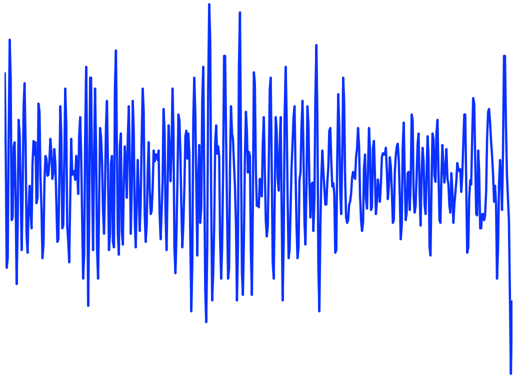
power distributions

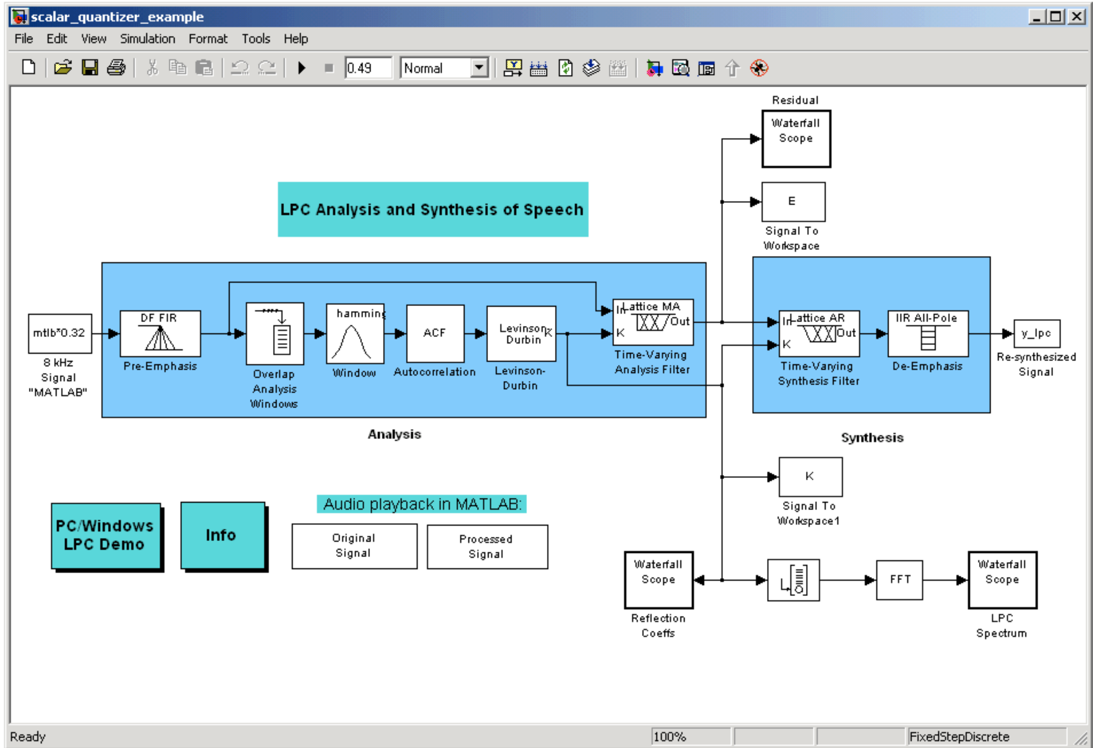


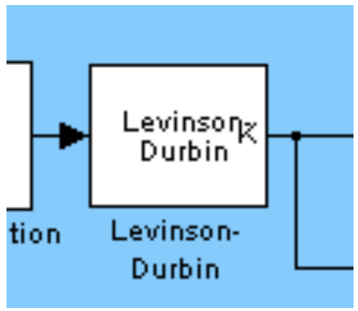
$(f_1 - f_2)$ is not a "signal"

Communications

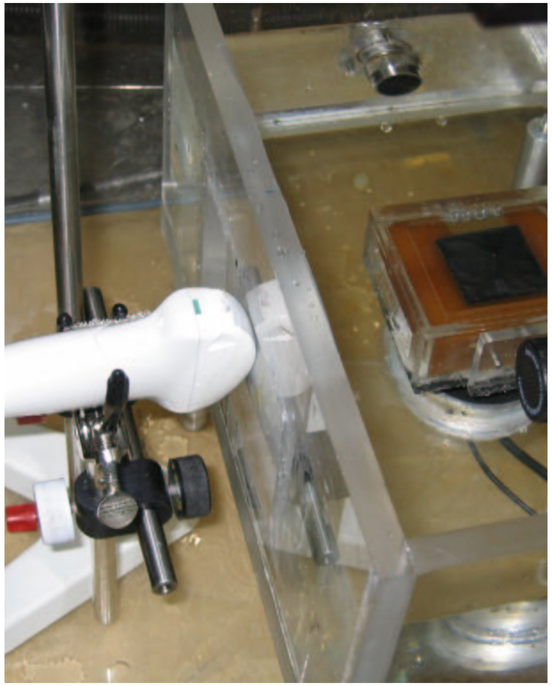
Speech analysis/coding



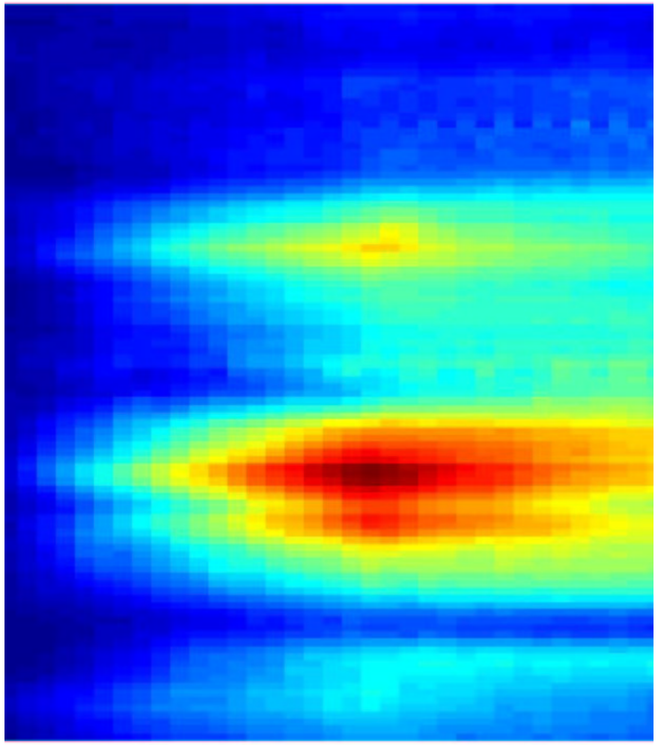




Medical diagnostics



Noninvasive temperature sensing

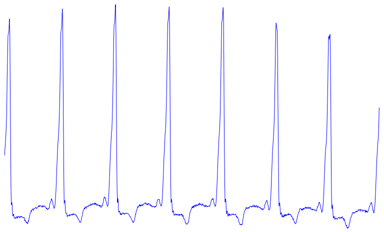


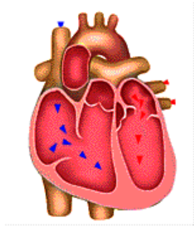
Temperature field

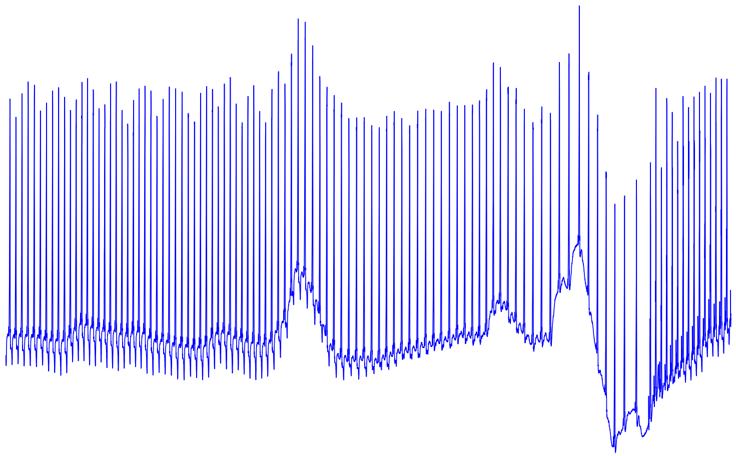
with E. Ebbini & A.N. Amini

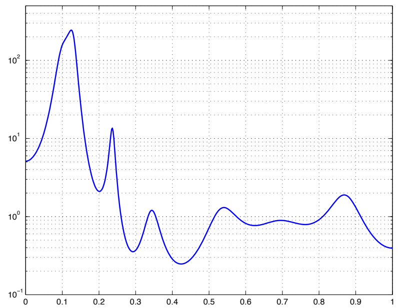
In IEEE Trans. on Biomedical Engineering, 2005

Medical diagnostics

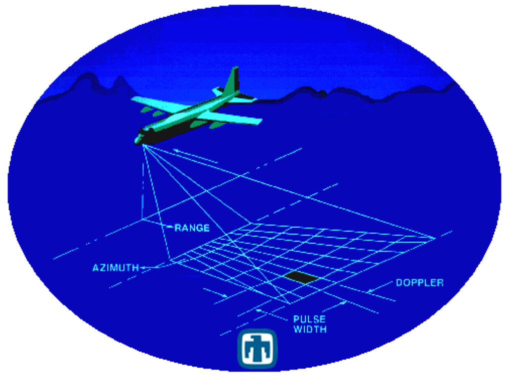




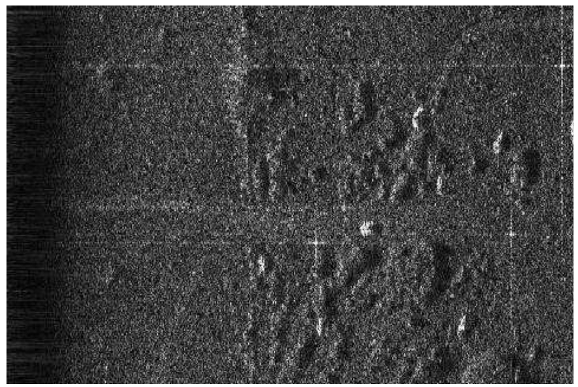




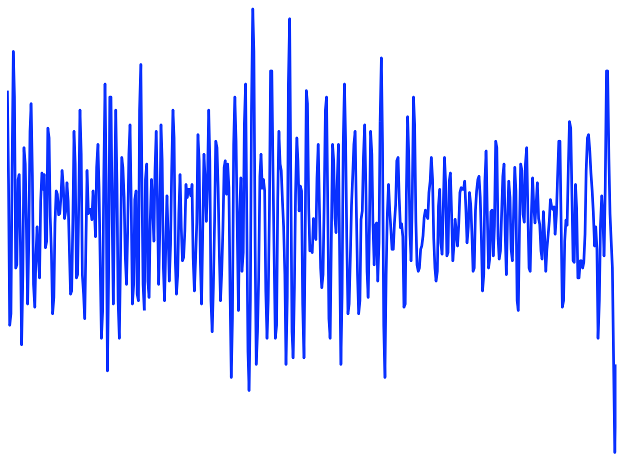
Radar (SAR)



http://www.sandia.gov/radar/images/3dsar.gif

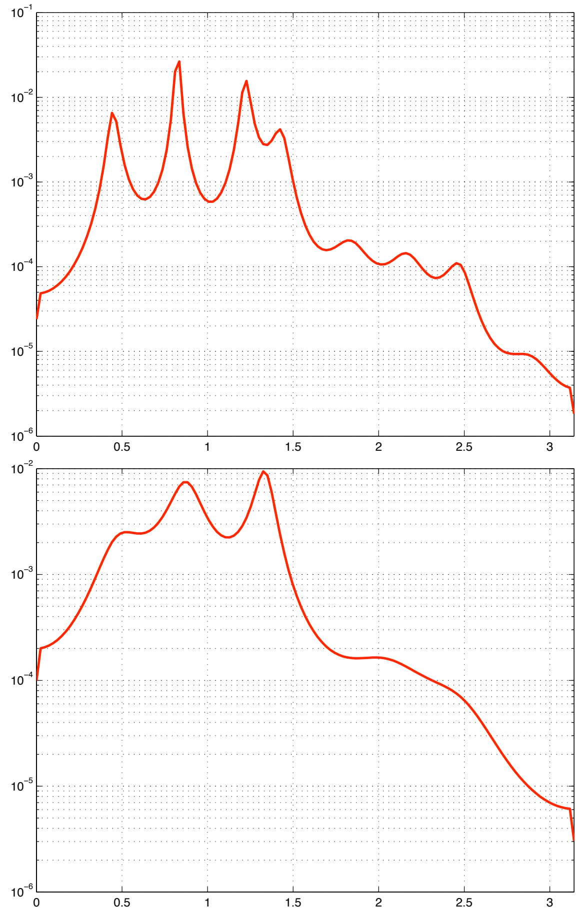


Quantitative analysis



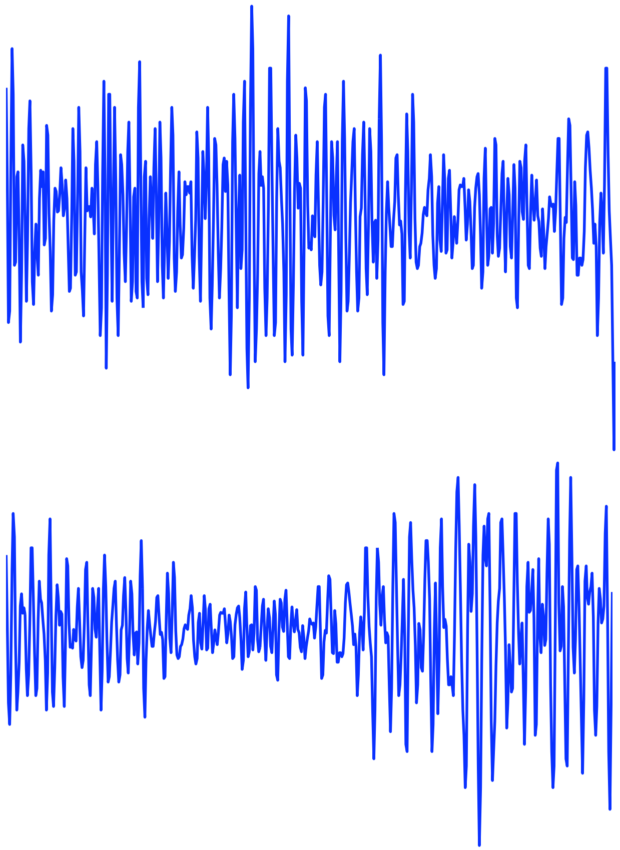
different

methods



How can we compare power spectra?

Quantitative analysis

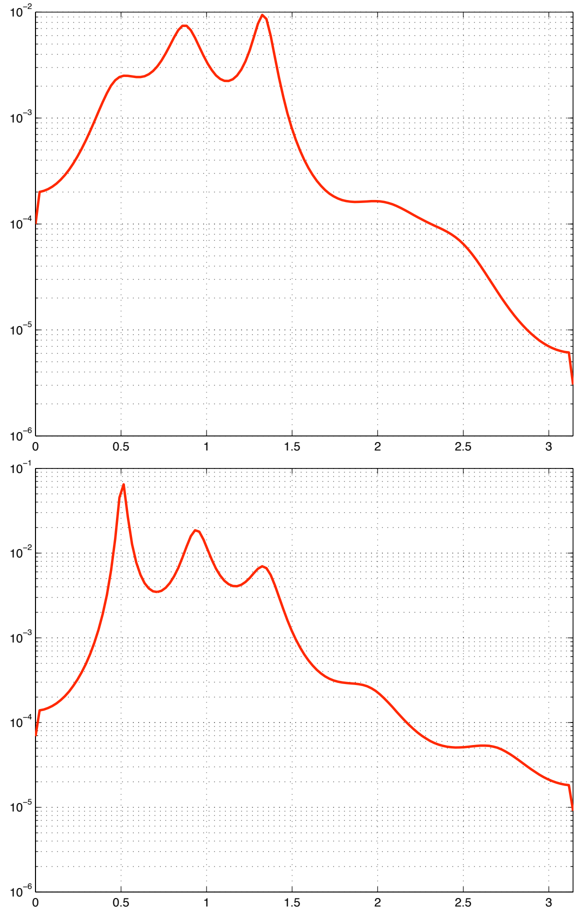


→

same

method

→



How can we compare power spectra?

How can we compare power spectra?

Question:
what is a natural notion of distance between power spectral densities?

quantify uncertainty

signal classification, detect structural changes

system identification, tune algorithms, sensor technology

Plan of the talk

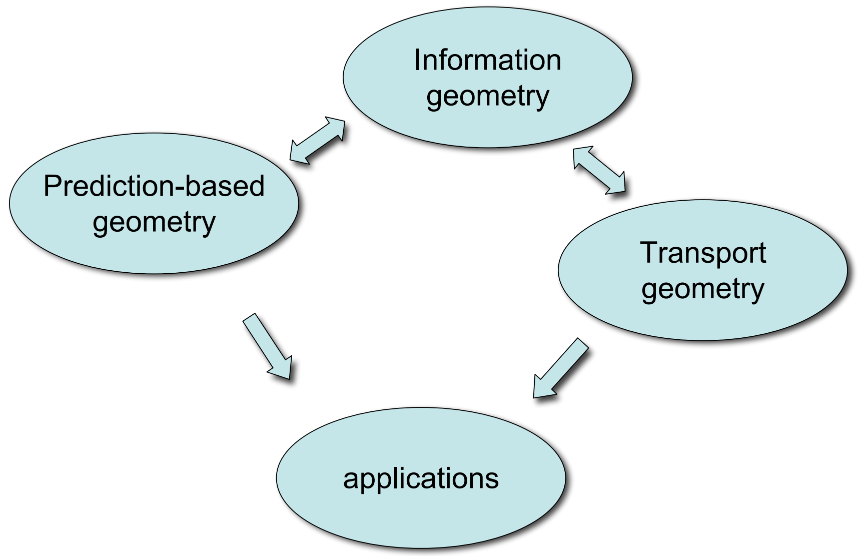
Metrics based on

prediction theory (Szegö, Kolmogorov)

parallels with information geometry (Fisher, Rao)

transport geometry (Monge-Kantorovich)

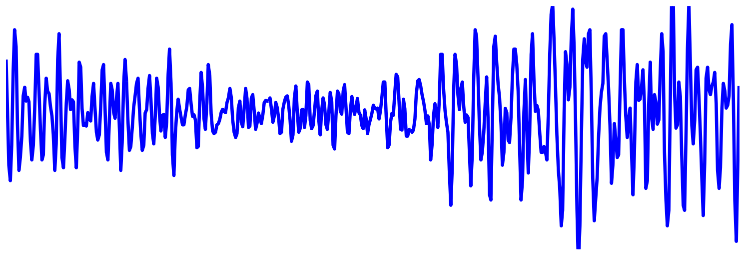
Case studies & applications

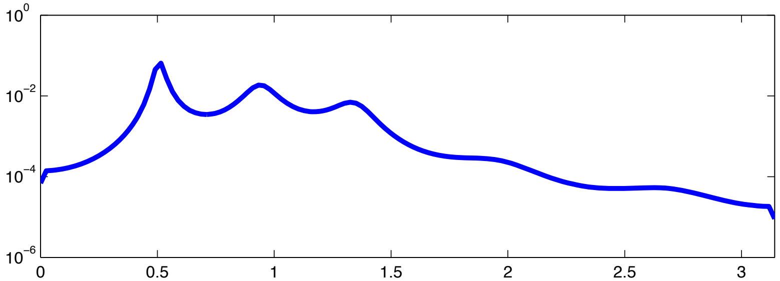


Setting

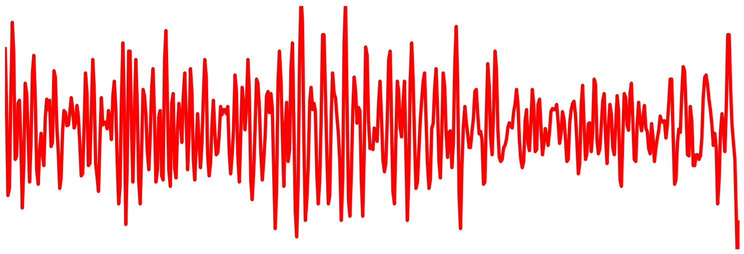
$\ldots u_{-1}, u_0, u_1, u_2, \ldots$

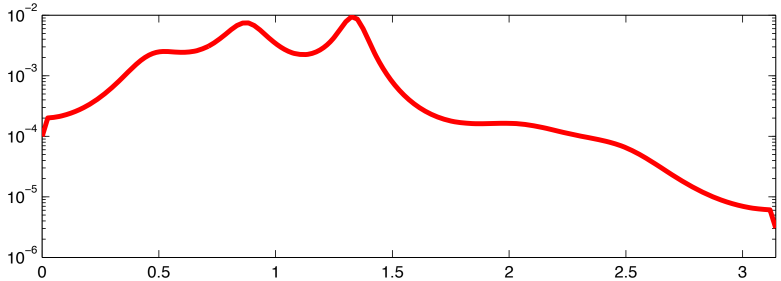
$\ldots u_{-1}, u_0, u_1, u_2, \ldots$





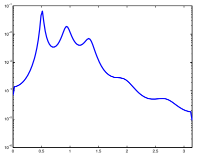
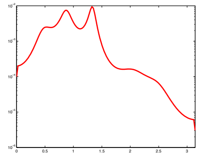
$f_1(\theta)$





$f_2(\theta)$

What is it we would like to have?

distance( , )

$f_1(\theta)$, $f_2(\theta)$

- metric
- meaningful & natural

candidates?

Kullback-Leibler, Bregman, Itakura-Saito, Makhoul,..

convex functionals

perceptual qualities

Linear prediction

One-step-ahead prediction: $u_{\text{present}} - \hat{u}_{\text{present}|\text{past}}$

with $\hat{u}_{\text{present}|\text{past}} := \sum_{\text{past}} \alpha_k u_k$

$$E\{|u_{\text{present}} - \hat{u}_{\text{present}|\text{past}}|^2\} = \text{variance of prediction error}$$

Szegö's theorem

One-step-ahead prediction:

$$\text{least error variance} \quad = \quad \exp\left\{\frac{1}{2\pi}\int \log f(\theta)d\theta\right\}$$

G. Szegö

it is a geometric mean. . .

$$\exp\{\frac{1}{3}(\log f_1 + \log f_2 + \log f_3)\} = \sqrt[3]{f_1 f_2 f_3}$$

Degradation of prediction error variance

Use f_2 to design a predictor (assuming $u_{f_2,\text{time}}$).

Then compare how this performs on $u_{f_1,\text{time}}$ against the optimal based on f_1.

$$\frac{\overbrace{E\{|u_{f_1,\text{present}} - \sum_{\text{past}} a_{f_2,\text{past}}\, u_{f_1,\text{past}}|^2\}}^{\text{degraded variance}} - \text{optimal variance}}{\text{optimal variance}} \geq 0$$

Degradation of prediction variance

$$\frac{\overbrace{E\{|u_{f_1,\text{present}} - \sum_{\text{past}} a_{f_2,\text{past}}\, u_{f_1,\text{past}}|^2\}}^{\text{degraded variance}}}{\text{optimal variance}} = \frac{\text{arithmetic mean of } \left(\frac{f_1}{f_2}\right)}{\text{geometric mean of } \left(\frac{f_1}{f_2}\right)}$$

$$= \frac{\left(\frac{1}{2\pi}\int(\frac{f_1}{f_2})d\theta\right)}{\exp\left(\frac{1}{2\pi}\int\log(\frac{f_1}{f_2})d\theta\right)}$$

arithmetic over *geometric* mean (≥ 1)

Riemannian metric

$$\begin{aligned} f_1 &= f, \\ f_2 &= f + \Delta \end{aligned}$$

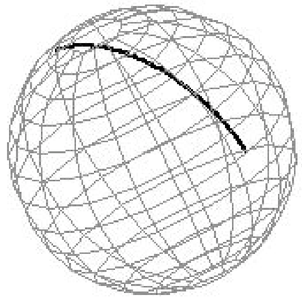
$$\frac{\overbrace{E\{|u_{f_1,\text{present}} - \sum_{\text{past}} a_{f_2,\text{past}}\, u_{f_1,\text{past}}|^2\}}^{\text{degraded variance}} - \text{optimal variance}}{\text{optimal variance}} \simeq$$

$$\delta(f, f+\Delta) = \frac{1}{2\pi}\int \left(\frac{\Delta}{f}\right)^2 d\theta - \left(\frac{1}{2\pi}\int \left(\frac{\Delta}{f}\right) d\theta\right)^2$$

variance-like: (mean square) - (arithmetic-mean)^2

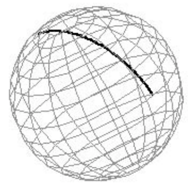
Geodesics

Paths $f_{\mathbf{r}}$ ($\mathbf{r} \in [0,1]$) between f_0, f_1 of minimal length $\int_0^1 \sqrt{\delta\left(f_{\mathbf{r}}, f_{\mathbf{r}+d\mathbf{r}}\right)}$



each point represents a different power spectral density

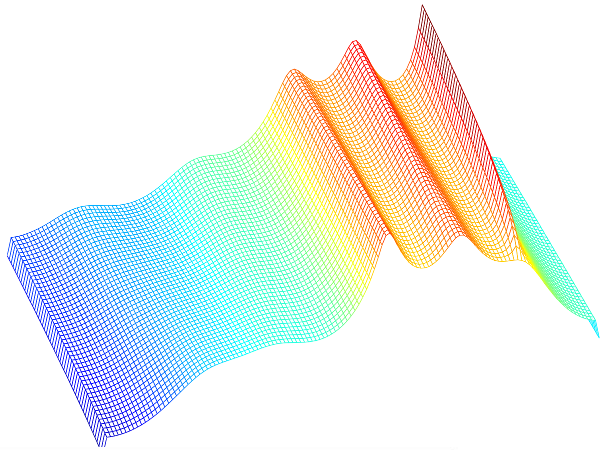
Geodesics

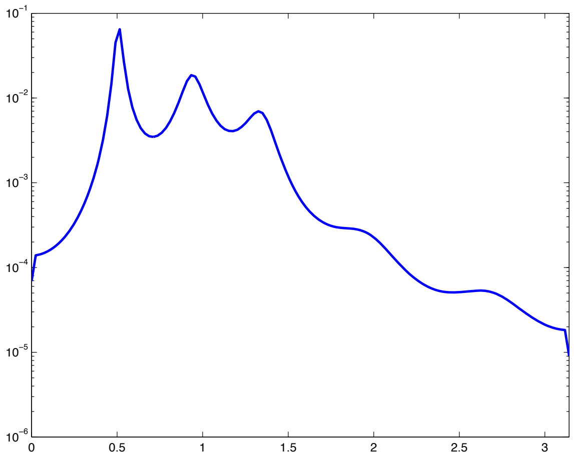


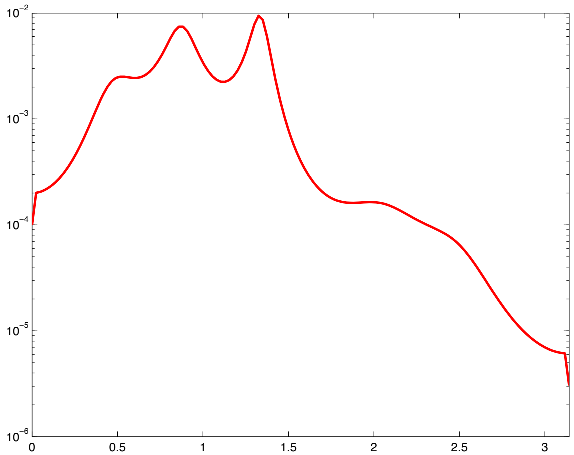
The geodesics are exponential families:

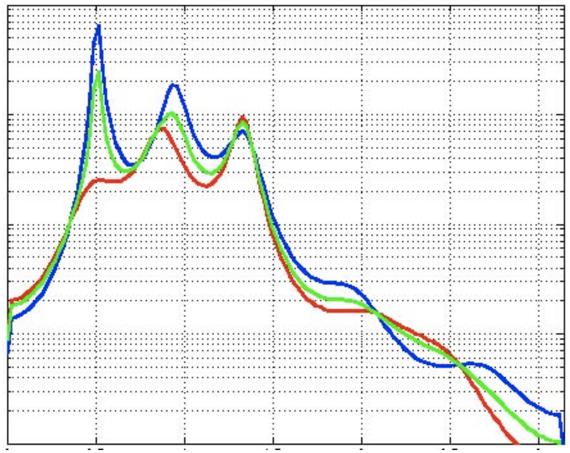
$$f_{\mathbf{r}} = f_0 \left(\frac{f_1}{f_0} \right)^{\mathbf{r}}, \quad \mathbf{r} \in [0, 1]$$

$$= \exp\{ \, (1 - \mathbf{r}) \log (f_0) + \mathbf{r} \log (f_1) \, \}$$



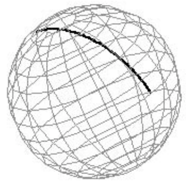






morphing

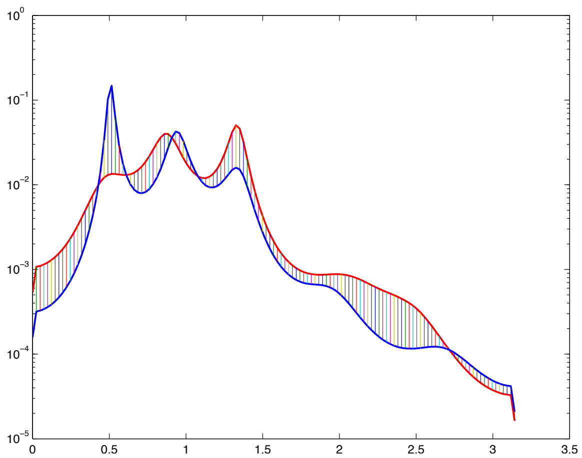
Geodesic distance: metric

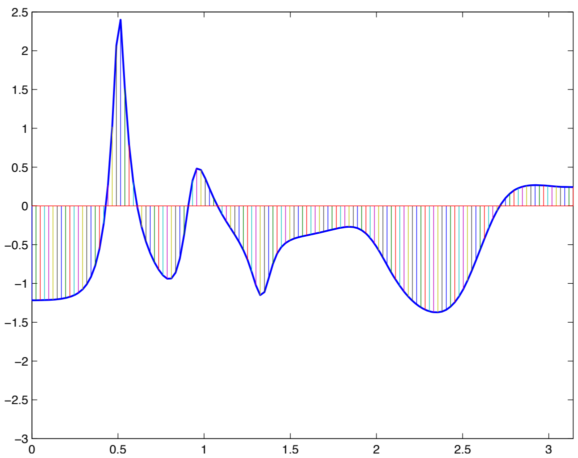


The path-length is

$$d(f_0, f_1) := \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\log\frac{f_1}{f_0}\right)^2 d\theta - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left(\frac{f_1}{f_0}\right) d\theta\right)^2}$$

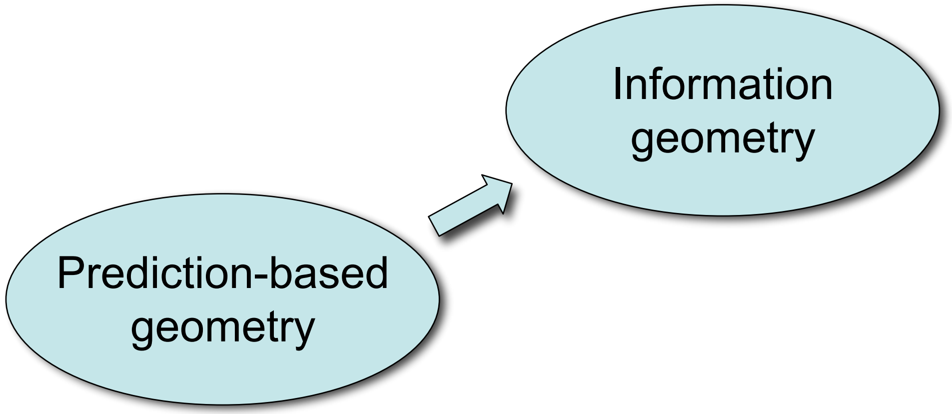
variance-like distance on logarithms: (mean square) - (arithmetic-mean)^2
scale-insensitive, “shape” recognizer





$$\log\frac{f_1}{f_0} \quad = \quad \log(f_1) - \log(f_0)$$

In IEEE Trans. on Signal Processing, Aug. 2007


Information geometry
Prediction-based geometry

Information geometry – *parallels*

Information geometry

$f \rightsquigarrow \mathbf{p}$: probability density

$$I = E_{\mathbf{p}}\{(\partial_\lambda \log \mathbf{p}_\lambda)^2\} \ \ \delta\lambda^2$$

Fisher information metric

$$I = \sum \frac{\Delta^2}{\mathbf{p}}$$

R. Fisher

C.R. Rao

Information geometry – *parallels*

R. Leibler

S. Kullback

Expected "message-length increase":

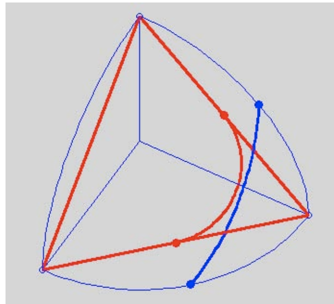
$$H(\mathbf{p}_1|\mathbf{p}_0) \;=\; \Big(-\sum \mathbf{p}_1 \log(\mathbf{p}_0)\Big) - \Big(-\sum \mathbf{p}_1 \log(\mathbf{p}_1)\Big)$$

Fisher information metric

$$\mathbf{p}_0 = \mathbf{p}$$
$$\mathbf{p}_1 = \mathbf{p} + \Delta$$

$$I = \sum \frac{\Delta^2}{\mathbf{p}}$$

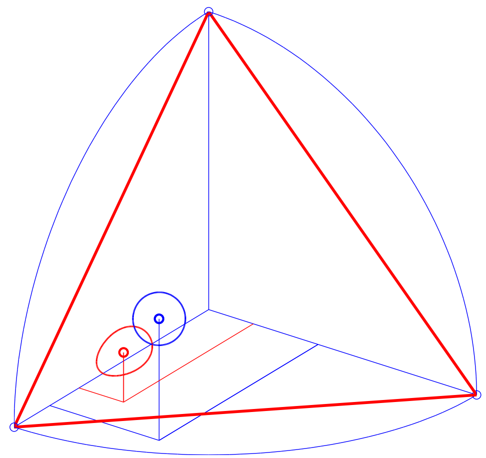
Information geometry – *parallels*



Geodesics: great circles

$$\mathbf{p} \mapsto \sqrt{\mathbf{p}} \in \text{Sphere}$$

$$\begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} \mapsto \begin{pmatrix} \sqrt{p(1)} \\ \sqrt{p(2)} \\ \sqrt{p(3)} \end{pmatrix}$$



Geodesic distance: Arclength
Battacharyya distance

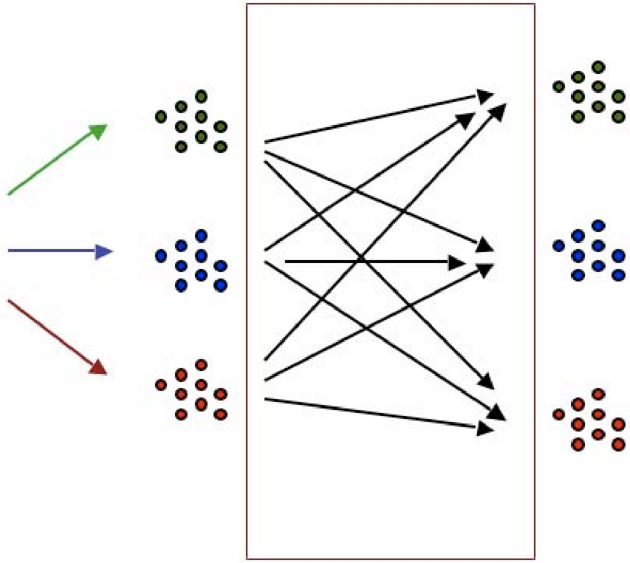
Information vs. prediction-based

$$\sum \frac{\Delta^2}{\mathbf{p}} \qquad \text{vs.} \qquad \int \left(\frac{\Delta}{\mathbf{f}}\right)^2 - \left(\int \frac{\Delta}{\mathbf{f}}\right)^2$$

$$\mathbf{p} \mapsto \sqrt{\mathbf{p}} \qquad \text{vs.} \qquad \mathbf{f} \mapsto \log \mathbf{f}$$

great circles vs. **logarithmic families**

Information geometry – *parallels*



Ability to differentiate decreases

$$\begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix} \mapsto \mathrm{M} \begin{pmatrix} p(1) \\ p(2) \\ p(3) \end{pmatrix}$$

Chentsov's theorem:

Stochastic maps are contractive

under *Fisher metric*

and

Fisher metric is the unique Riemannian metric with this property

What is the analog for power spectra?

addititive noise

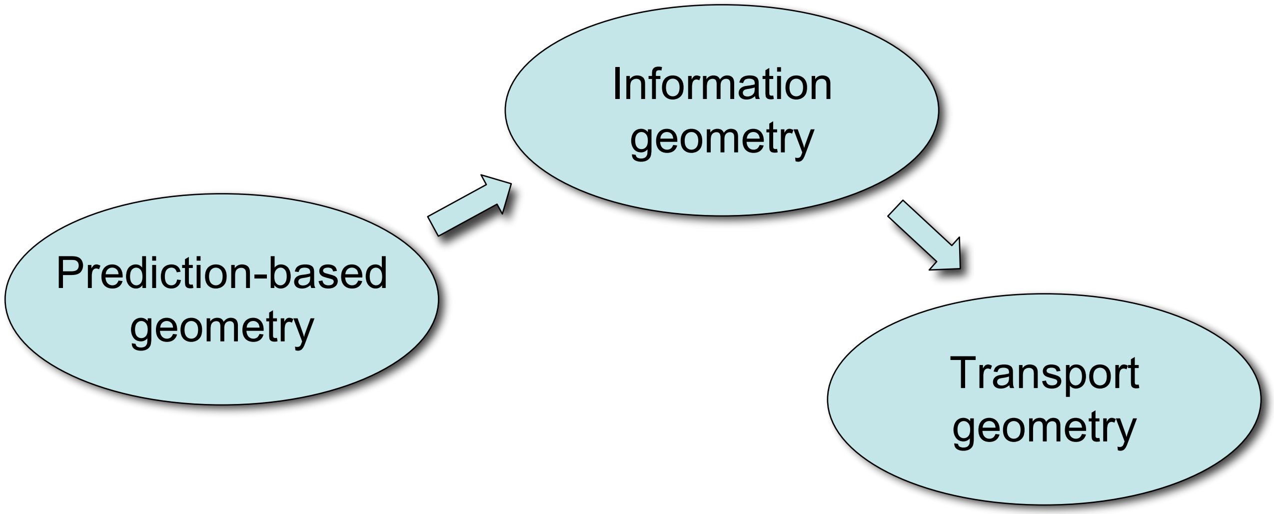
$f \ \mapsto f + f_{\text{noise}}$

multiplicative noise

$f \ \mapsto f \star f_{\text{noise}}$

continuity of moments (second-order statistics)

$f \ \mapsto \text{integrals of } f$


Information geometry
Prediction-based geometry
Transport geometry

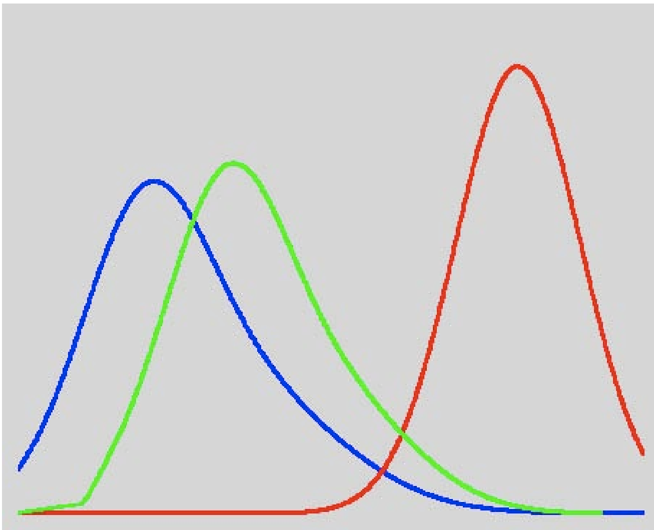
Transport geometry

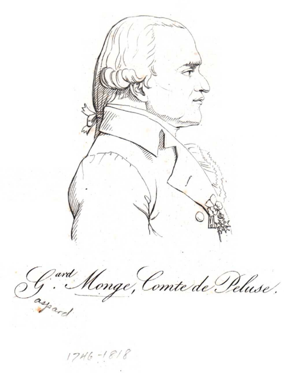
Transport geometry

Monge-Kantorovich problem

minimize cost of transferring mass

$$\int \text{cost}(x \rightarrow y) \; \times \; \text{mass}(dx, dy)$$





G. Monge

L. Kantorovich

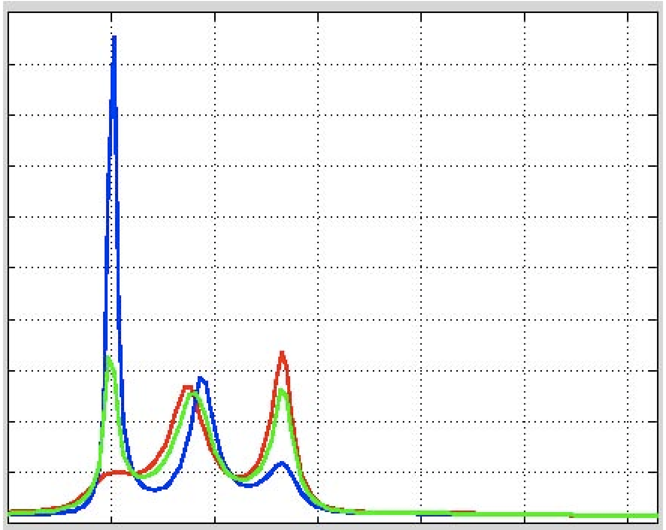
Transport for power spectra

Transport-based metric

distances do not increase

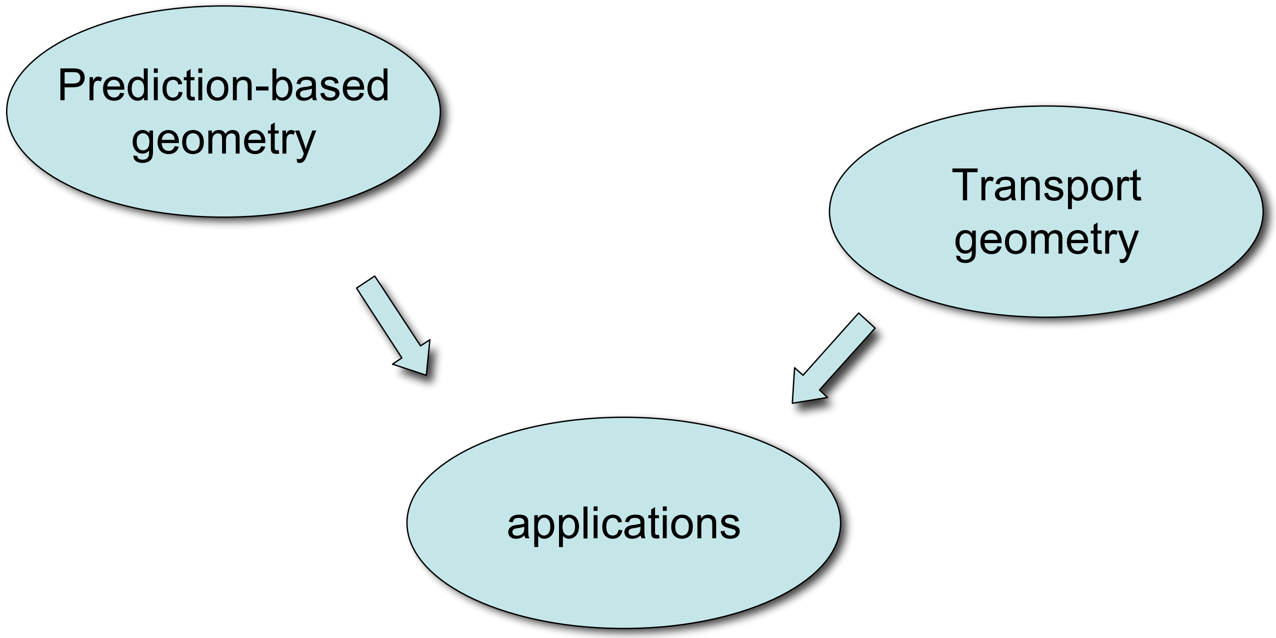
under additive noise
and multiplicative noise
with power ≤ 1

$+$ continuity of statistics



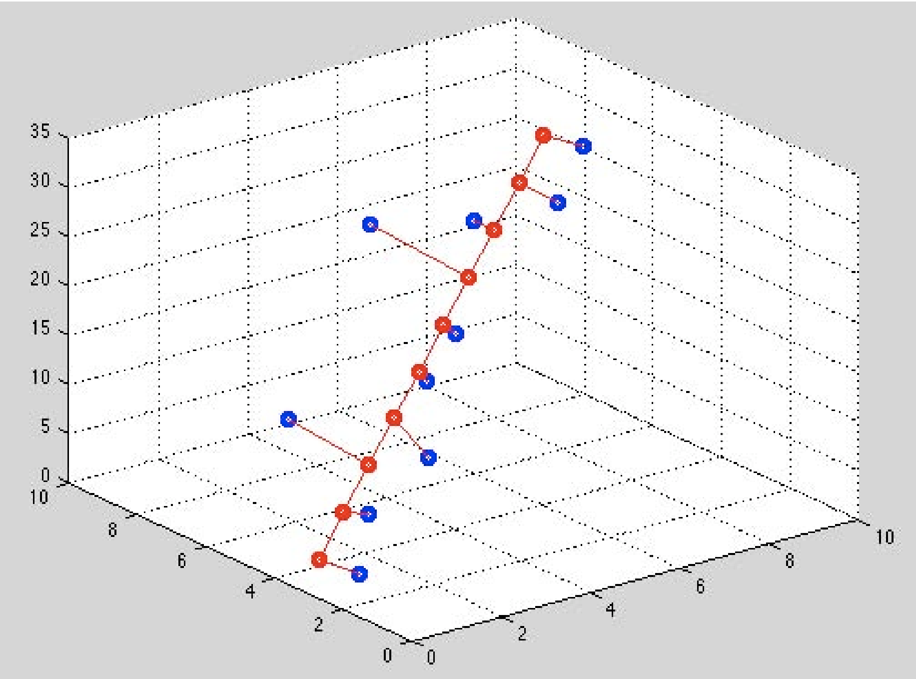
$$\mathbf{metric} \;=\; \min\left(\text{cost of transport}(\hat{f}_0, \hat{f}_1) \;+\; \text{normalization}\right)$$

with Johan Karlsson (KTH) & Mir Shahrouz Takyar


Prediction-based geometry
Transport geometry
applications

Fitting geodesics

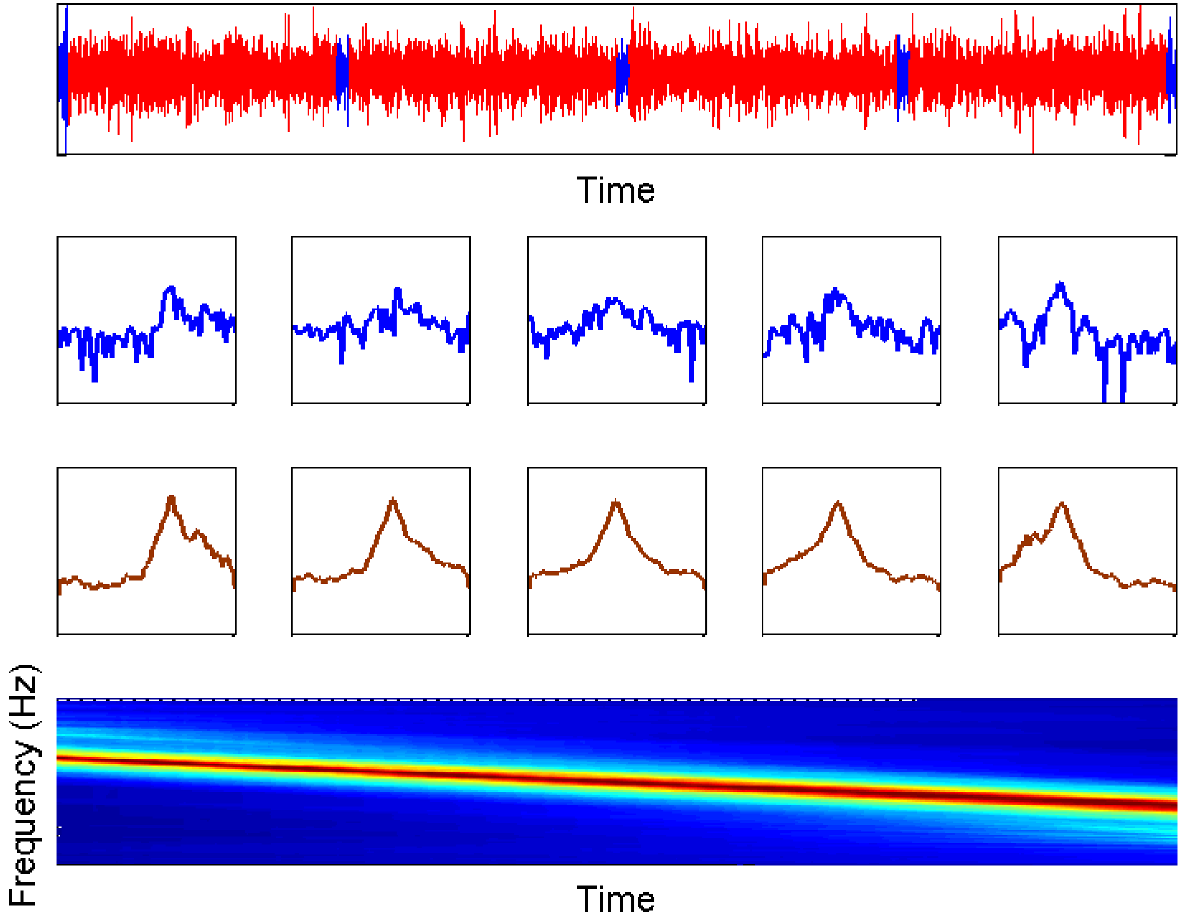
applications

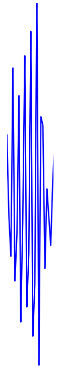


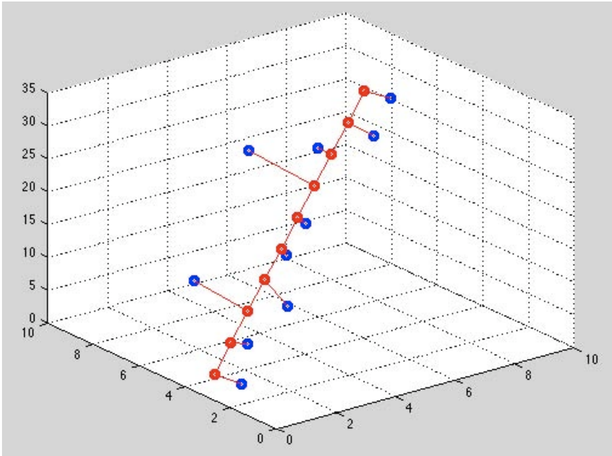
K.F. Gauss

Least squares: The theory of motion of heavenly bodies, Gauss, K.F.

Tracking with geodesics





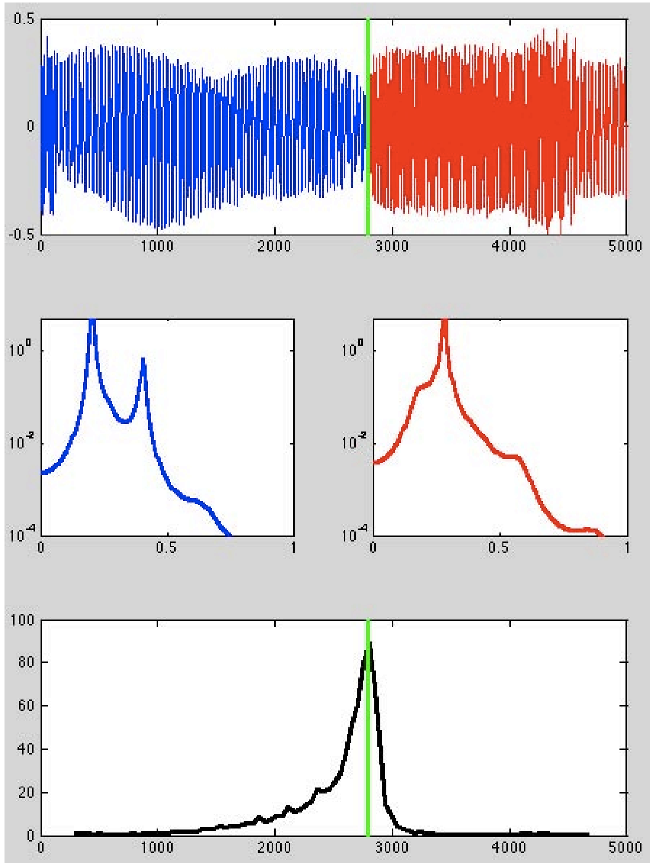


with Xianhua Jiang

Voice & sounds

John Weissmuller's MGM Tarzan Yell

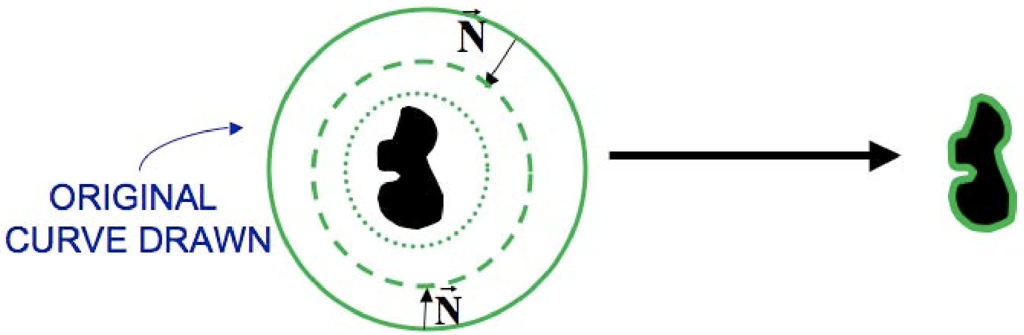
http://www.complxmind.com

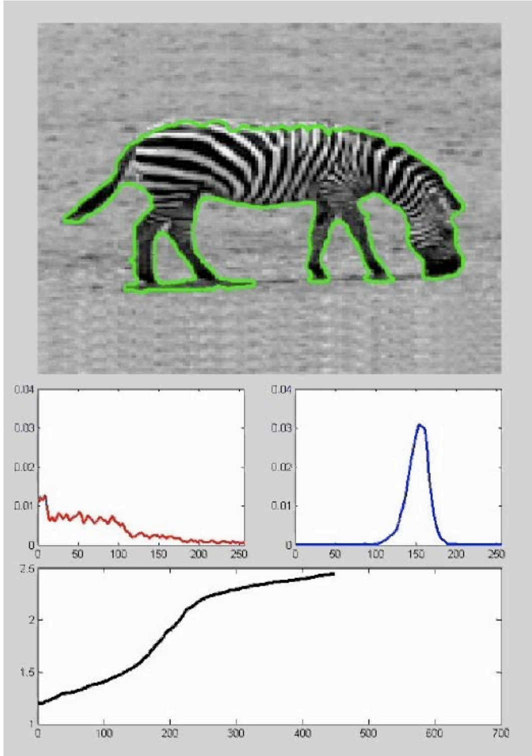


Images & more

Geometric active contours

$$\frac{\partial}{\partial t}\mathrm{Curve} = \nabla_{\mathrm{Curve}}\ \mathrm{metric}(f_{\mathrm{inside}}, f_{\mathrm{outside}})$$





with Romeil Sandhu and Allen Tannenbaum

Images & more

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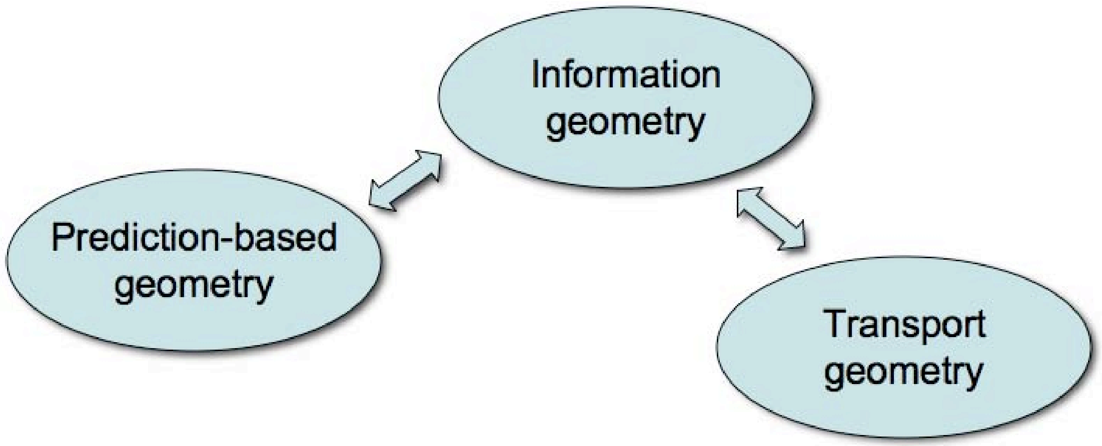
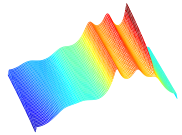
Concluding thoughts

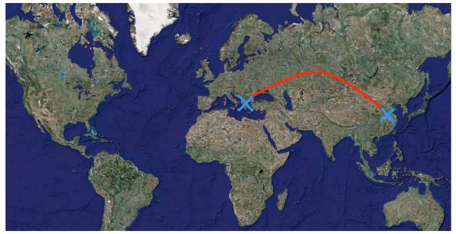
Metrics
in spectral analysis

- Operational significance
- Effect of natural transformations



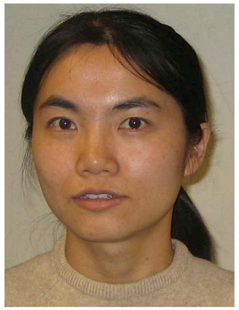
Thank you for your attention





thanks to

Xianhua Jiang



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