Contraction Theory in Systems and Control

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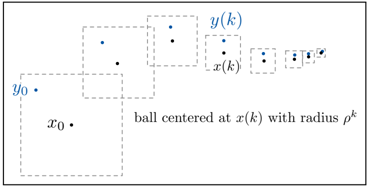

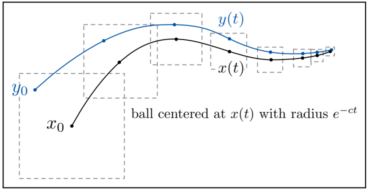

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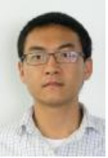
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contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

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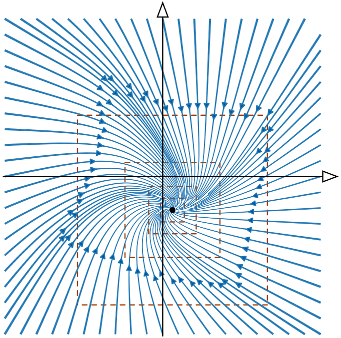
highly-ordered transient and asymptotic behavior:

1. unique globally exponential stable equilibrium
   & two natural Lyapunov functions
2. robustness properties
   - bounded input, bounded output (iss)
   - finite input-state gain
   - robustness margin wrt unmodeled dynamics
   - robustness margin wrt delayed dynamics
3. periodic input, periodic output
4. modularity and interconnection properties
5. accurate numerical integration and equilibrium point computation



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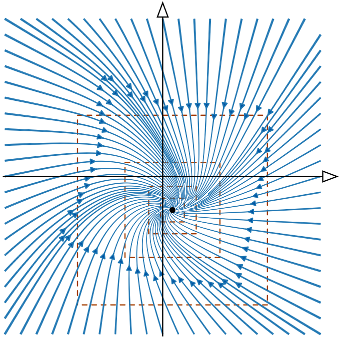

search for contraction properties

design engineering systems to be contracting

Contraction theory: historical notes

- **Origins**

  S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922.

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  S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958. URL **http://mi.mathnet.ru/eng/ivm2980**. (in Russian)

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- **Systems and control**:

  W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6): 683–696, 1998.

- **Incomplete list of contributors who influenced me**
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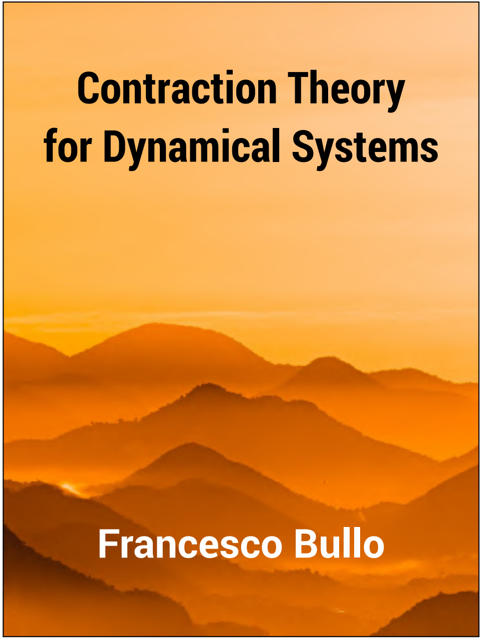
- **Surveys**:

Z. Aminzare and E. D. Sontag. Contraction methods for nonlinear systems: A brief introduction and some open problems. In *IEEE Conf. on Decision and Control*, pages 3835–3847, Dec. 2014.

M. Di Bernardo, D. Fiore, G. Russo, and F. Scafuti. Convergence, consensus and synchronization of complex networks via contraction theory. In *Complex Systems and Networks*. Springer, 2016.

H. Tsukamoto, S.-J. Chung, and J.-J. E. Slotine. Contraction theory for nonlinear stability analysis and learning-based control: A tutorial overview. *Annual Reviews in Control*, 52:135–169, 2021.

P. Giesl, S. Hafstein, and C. Kawan. Review on contraction analysis and computation of contraction metrics. *Journal of Computational Dynamics*, 10(1):1–47, 2023.



Contraction Theory for Dynamical Systems, Francesco Bullo, KDP, 1.0 edition, 2022, ISBN 979-8836646806

1. Textbook with exercises and answers. Format: textbook, slides, and paperbook
2. Content:
   Fixed point theory
   Theory of contracting dynamics on vector spaces
   Applications to nonlinear and interconnected systems
3. Self-Published and Print-on-Demand at:
   https://www.amazon.com/dp/B0B4K1BTF4
4. PDF Freely available at
   http://motion.me.ucsb.edu/book-ctds
5. 10h minicourse on youtube:
   https://youtu.be/RvR47ZbqJjc
6. Future version to include: systems on Riemannian manifolds, homogeneous spaces, and solid cones

   "Continuous improvement is better than delayed perfection"
   Mark Twain

Outline

Linear algebra: induced norms

Vector norm	Induced matrix norm	Induced matrix log norm
$\|x\|_1 = \sum_{i=1}^n \lvert x_i\rvert$	$\|A\|_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n \lvert a_{ij}\rvert$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1,i\neq j}^n \lvert a_{ij}\rvert\right)$ $=$ max column "absolute sum" of A
$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$	$\|A\|_2 = \sqrt{\lambda_{\text{max}}(A^\top A)}$	$\mu_2(A) = \lambda_{\text{max}}\left(\frac{A+A^\top}{2}\right)$
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Discrete-time dynamics and Lipschitz constants

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$$\begin{aligned}\mathsf{Lip}(\mathsf{F}) &= \inf\{\ell > 0 \text{ such that } \|\mathsf{F}(x) - \mathsf{F}(y)\| \leq \ell\|x - y\| \quad \text{for all } x, y\} \\ &= \sup_x \|\mathsf{J}_{\mathsf{F}}(x)\|\end{aligned}$$

For **scalar map** f, $\mathsf{Lip}(f) = \sup_x |f'(x)|$

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$$\begin{aligned}\|x\|_{2,P} &= (x^\top P x)^{1/2} & \mathsf{Lip}_{2,P}(\mathsf{F}_A) &= \|A\|_{2,P} \leq \ell & \iff \quad & A^\top P A \preceq \ell^2 P \\ \|x\|_{\infty,\eta} &= \max_i |x_i|/\eta_i & \mathsf{Lip}_{\infty,\eta}(\mathsf{F}_A) &= \|A\|_{\infty,\eta} \leq \ell & \iff \quad & \eta^\top |A| \leq \ell \eta^\top\end{aligned}$$

Banach contraction theorem for discrete-time dynamics:

If $\rho := \text{Lip}(F) < 1$, then

1. F is **contracting** = distance between trajectories decreases exp fast (ρ^k)
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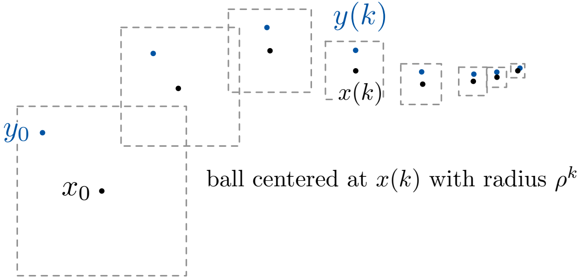

From discrete to continuous time

The **induced log norm** of $A \in \mathbb{R}^{n \times n}$ wrt to $\|\cdot\|$:

$$\mu(A) := \lim_{h \to 0^+} \frac{\|I_n + hA\| - 1}{h}$$

subadditivity: $\mu(A + B) \leq \mu(A) + \mu(B)$

scaling: $\mu(bA) = b\mu(A), \quad \forall b \geq 0$

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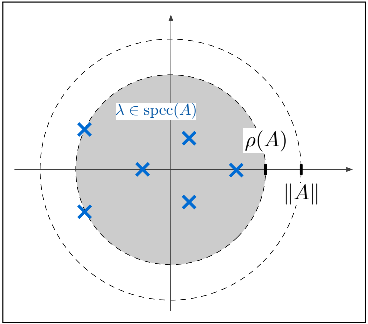

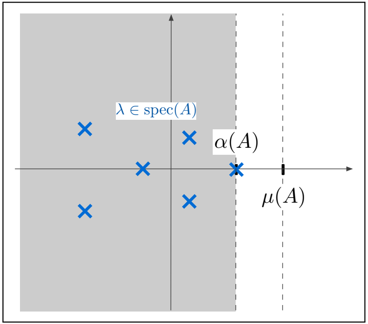

Example induced log norms

Vector norm	Induced matrix norm	Induced matrix log norm
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Continuous-time dynamics and one-sided Lipschitz constants

$\dot{x} = \mathsf{F}(x)$ on $\mathbb{R}^n$ with norm $\|\cdot\|$ and induced log norm $\mu(\cdot)$

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One-sided Lipschitz constant

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For **scalar map** f, $\quad \mathrm{osLip}(f) = \sup_x f'(x)$

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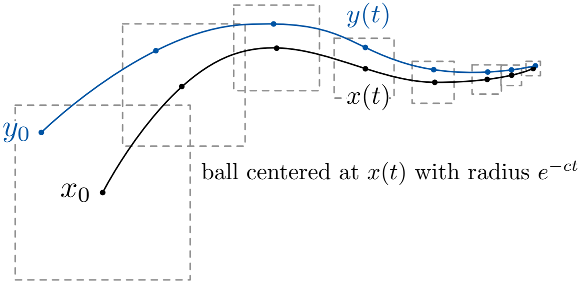

From inner products to weak pairings

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_2^2 \;=\; \dot{x}^\top x \;=\; \langle\!\langle \dot{x}, x \rangle\!\rangle$$

$$\implies \qquad \tfrac{1}{2}D^+\|x(t)\|^2 \;=:\; [\![\dot{x}, x]\!]$$

- D^+ is upper-right Dini derivative

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- D^+ is upper-right Dini derivative
- **weak pairing** $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ exists for each norm, i.e.,

$$[\![y, x]\!]_1 \;:=\; \|x\|_1 \operatorname{sign}(x)^\top y \qquad \text{(sign pairing)}$$

$$[\![y, x]\!]_\infty \;:=\; \max_{i \in \mathcal{A}_\infty(x)} x_i y_i \qquad \text{for } \mathcal{A}_\infty(x) = \{i \mid |x_i| = \|x\|_\infty\} \qquad \text{(max pairing)}$$

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theory of weak pairings: computational properties and applications to monotone operators

Log norm bounds	Demidovich conditions	One-sided Lipschitz conditions
$\mu_{2,P}(\mathsf{J_F}(x)) \leq -c$	$P\mathsf{J_F}(x) + \mathsf{J_F}(x)^\top P \preceq -2cP$	$(x-y)^\top P\big(\mathsf{F}(x) - \mathsf{F}(y)\big) \leq -c\\|x-y\\|_{P^{1/2}}^2$
$\mu_1(\mathsf{J_F}(x)) \leq -c$	$\operatorname{sign}(v)^\top \mathsf{J_F}(x)v \leq -c\\|v\\|_1$	$\operatorname{sign}(x-y)^\top(\mathsf{F}(x) - \mathsf{F}(y)) \leq -c\\|x-y\\|_1$
$\mu_\infty(\mathsf{J_F}(x)) \leq -c$	$\max_{i \in \mathcal{A}_\infty(v)} v_i\,(\mathsf{J_F}(x)v)_i \leq -c\\|v\\|_\infty^2$	$\max_{i \in \mathcal{A}_\infty(x-y)} (x_i-y_i)(\mathsf{F}_i(x)-\mathsf{F}_i(y)) \leq -c\\|x-y\\|_\infty^2$

Each row = three equivalent statements. To be understood for all $x, y \in \mathbb{R}^n$ and all $v \in \mathbb{R}^n$.

One sided Lipschitz conditions

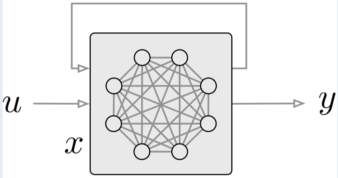
1. **simple sufficient condition** for uniqueness of continuous ODEs in: A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer, 1988. ISBN 902772699X (Chapter 1, page 5, citing Krasnosel'skii and Krein 1955)
2. **one-sided Lipschitz maps** in: E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer, 1993. (Section I.10)

3. **uniformly decreasing maps** in: L. Chua and D. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks. *IEEE Transactions on Circuits and Systems*, 23(6):355–379, 1976.
4. **maps with negative nonlinear measure** in: H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2): 360–370, 2001.
5. **dissipative Lipschitz maps** in: T. Caraballo and P. E. Kloeden. The persistence of synchronization under environmental noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461 (2059):2257–2267, 2005.
6. **maps with negative lub log Lipschitz constant** in: G. Söderlind. The logarithmic norm. History and modern theory. *BIT Numerical Mathematics*, 46(3):631–652, 2006.
7. **QUAD maps** in: W. Lu and T. Chen. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D: Nonlinear Phenomena*, 213(2):214–230, 2006.
8. **incremental quadratically stable maps** in: L. D'Alto and M. Corless. Incremental quadratic stability. *Numerical Algebra, Control and Optimization*, 3:175–201, 2013.

Advantages of non-Euclidean approaches

1. *well suited for certain class of systems*
   ℓ_1 for monotone flow systems
2. *computational advantages*
   ℓ_1/ℓ_∞ constraints lead to LPs, whereas ℓ_2 constraints leads to LMIs
3. *robustness to structural perturbations*
   ℓ_1/ℓ_∞ contractions are connectively robust (i.e., edge removal)
4. *adversarial input-output analysis*
   ℓ_∞ better suited for the analysis of adversarial examples than ℓ_2
5. *asynchronous distributed computation*
   ℓ_∞ contractions converge under fully asynchronous distributed execution

Application: ℓ_∞-contracting neural networks



$$\dot{x} = -x + \Phi(Ax + Bu + b) \qquad (\textit{recurrent NN})$$

$$x = \Phi(Ax + Bu + b) \qquad (\textit{implicit NN})$$

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu + b) \qquad (\textit{forward Euler})$$

If

$$\mu_\infty(A) < 1 \qquad \left(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\right)$$

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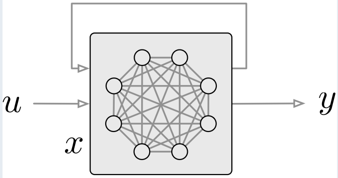

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If

$$\mu_\infty(A) < 1 \qquad \left(\text{i.e., } a_{ii} + \sum_j |a_{ij}| < 1 \text{ for all } i\right)$$

- recurrent NN is contracting with rate $1 - \mu_\infty(A)_+$
- implicit NN is well posed
- forward Euler is contracting with factor $1 - \dfrac{1-\mu_\infty(A)_+}{1-\min_i(a_{ii})_-}$ at $\alpha = \dfrac{1}{1-\min_i(a_{ii})_-}$

Detour: convexity and fixed point theory

For differentiable $V : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. V is **strongly convex** with parameter m
2. $-\,\mathrm{grad} V$ is m-**strongly infinitesimally contracting** with respect to $\| \cdot \|_2$

Forward Euler theorem for contracting dynamics

Given arbitrary norm $\|\cdot\|$, equivalent statements

1. $\dot{x} = \mathsf{F}(x)$ is infinitesimally contracting
2. there exists $\alpha > 0$ such that $x_{k+1} = x_k + \alpha \mathsf{F}(x_k)$ is contracting

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Given *contraction rate* c and *Lipschitz constant* ℓ, define *condition number* $\kappa = \dfrac{\ell}{c} \geq 1$

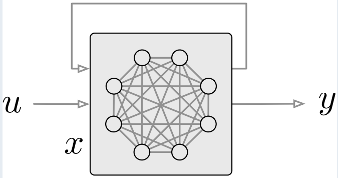
1. $\mathsf{Id} + \alpha \mathsf{F}$ is contracting for

$$0 < \alpha < \frac{1}{c\kappa(1+\kappa)}$$

2. the optimal step size minimizing and minimum contraction factor:

$$\alpha^* = \frac{1}{c}\Big(\frac{1}{2\kappa^2} - \frac{3}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)\Big)$$
$$\ell^* = 1 - \frac{1}{4\kappa^2} + \frac{1}{8\kappa^3} + \mathcal{O}\Big(\frac{1}{\kappa^4}\Big)$$

Application: ℓ_∞-contracting neural networks



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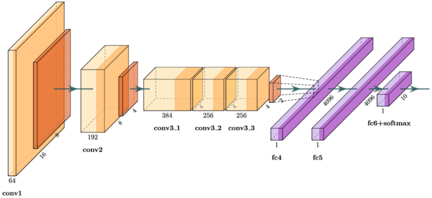
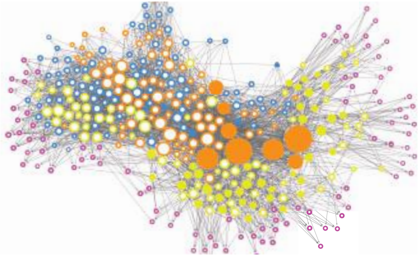
- recurrent NN is contracting with rate $1 - \mu_\infty(A)_+$
- implicit NN is well posed
- forward Euler is contracting with factor $1 - \dfrac{1-\mu_\infty(A)_+}{1-\min_i(a_{ii})_-}$ at $\alpha^* = \dfrac{1}{1-\min_i(a_{ii})_-}$

Motivation: ℓ_∞-contracting neural networks

While most ML architectures are feedforward,
biological neural networks are recurrent and recent interest for implicit ML architectures



artificial neural network AlexNet ’12



C. elegans connectome ’17

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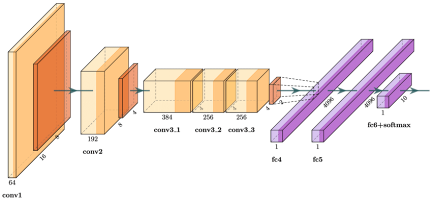

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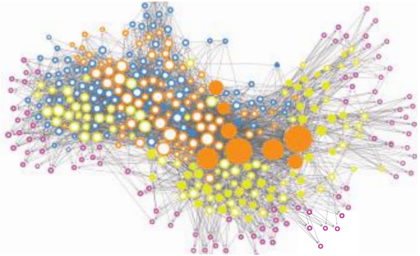

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For recurrent NN, ℓ_∞-contractivity characterizes the synaptic weights to ensure:

- reproducible & robust behavior
- highly-ordered transient+asymptotic dynamic behavior
- efficient computational methods

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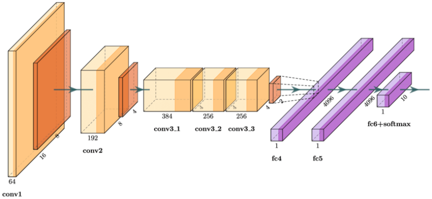

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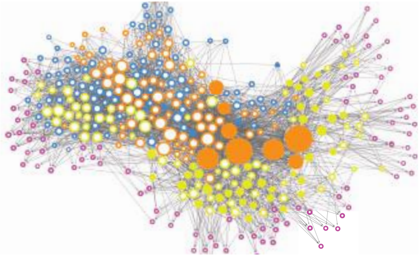

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For recurrent NN, ℓ_∞-contractivity characterizes the synaptic weights to ensure:

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A. Krizhevsky, I. Sutskever, and G. E. Hinton. Imagenet classification with deep convolutional neural networks. *Advances in Neural Information Processing Systems*, 25, 2012

G. Yan, P. E. Vértes, E. K. Towlson, Y. L. Chew, D. S. Walker, W. R. Schafer, and A.-L. Barabási. Network control principles predict neuron function in the Caenorhabditis elegans connectome. *Nature*, 550(7677):519–523, 2017.

Outline

$\#1$: From closed to open systems

Incremental ISS and input-state gain

Given normed spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, consider

$$\dot{x} = \mathsf{F}(x, u(t)), \qquad x_0 \in \mathcal{X}, \qquad u(t) \in \mathcal{U}$$

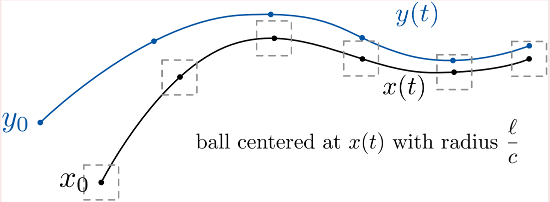
Assume:

- **contractivity wrt x**: $\quad \mathrm{osLip}_x(\mathsf{F}) \leq -c < 0,\quad$ uniformly in u
- **Lipschitz wrt u**: $\quad \mathrm{Lip}_u(\mathsf{F}) \leq \ell,\quad$ uniformly in x

Then

1. any soltns: $x(t)$ with input u_x and $y(t)$ with input u_y

$$D^+\|x(t)-y(t)\|_{\mathcal{X}} \leq -c\|x(t)-y(t)\|_{\mathcal{X}} + \ell\|u_x(t)-u_y(t)\|_{\mathcal{U}}$$



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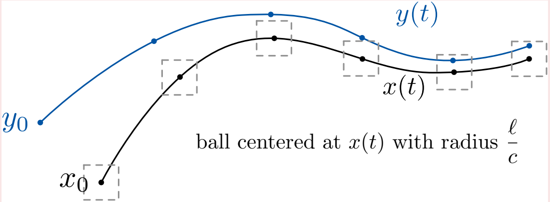


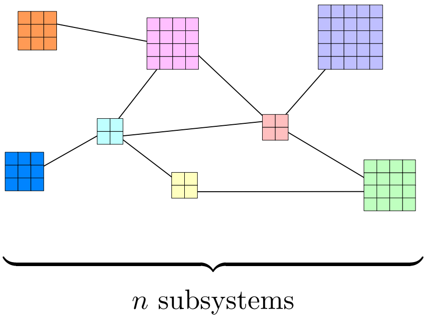
2. F is **incrementally ISS**, that is, for all x_0, y_0

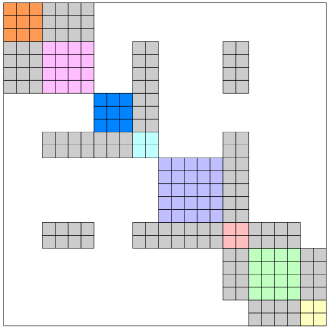
$$\|x(t)-y(t)\|_{\mathcal{X}} \;\le\; e^{-ct}\|x_0-y_0\|_{\mathcal{X}} \;+\; \frac{\ell(1-e^{-ct})}{c}\sup_{\tau\in[0,t]}\|u_x(\tau)-u_y(\tau)\|_{\mathcal{U}}$$

Consider n interconnected subsystems

$$\dot{x}_i = \mathsf{F}_i(x_i, x_{-i}), \qquad \text{for } i \in \{1, \dots, n\}$$

with state $x_i \in \mathbb{R}^{N_i}$
with states of connected subsystems $x_{-i} \in \mathbb{R}^{N-N_i}$, and
consider n *local norms* $\|\cdot\|_i$ on $\mathbb{R}^{N_i}$





Assume for each node i:

- **contractivity wrt x_i**: $\mathsf{osLip}_{x_i}(\mathsf{F}_i) \leq -c_i < 0$, uniformly in x_{-i}
- **Lipschitz wrt x_j**: $\mathsf{Lip}_{x_j}(\mathsf{F}_i) \leq \ell_{ij}$, uniformly in x_{-j}

Assume for each node i:

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Network contraction theorem

If the Lipschitz constants matrix $\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$ is **Hurwitz**

$\implies$ the **interconnected system** is infinitesimally contracting

History: interconnection of stable systems, method of vector Lyapunov functions, connective stability via M-matrix theory – Matrosov and Bellman 1962, Ström, Siljak, Russo/DiBernardo/Sontag, ...

$$\begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix} \text{ is } \textbf{Metzler}$$

Hurwitzness depends upon both topology and edge weights

- Hurwitz iff there exists a positive ξ such that $M\xi < \mathbb{0}_n$ (power method)
- Hurwitz iff Lyapunov diagonally stable

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- for $n = 2$, Hurwitz if and only if **small gain condition**

$$\text{cycle gain} := \frac{\ell_{12}}{c_1}\frac{\ell_{21}}{c_2} < 1$$

and, for $n \geq 3$, **network small-gain theorem for Metzler matrices**

X. Duan, S. Jafarpour, and F. Bullo. Graph-theoretic stability conditions for Metzler matrices and monotone systems. *SIAM Journal on Control and Optimization*, 59(5):3447–3471, 2021.

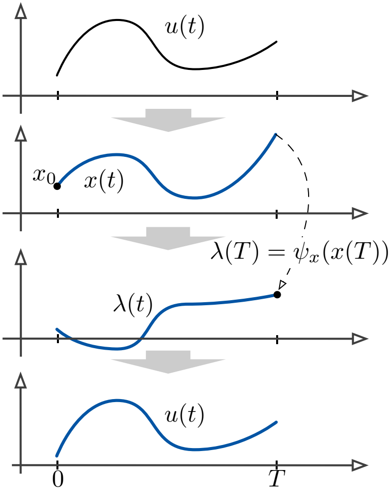
#3: From closed to systems with optimal controls

For $\dot{x} = \mathsf{F}(x, u)$, compute $u : [0, T] \to \mathbb{R}^k$ to minimize $\psi(x(T)) + \int_0^T \phi(x, u) dt$

#3: From closed to systems with optimal controls

For $\dot{x} = \mathsf{F}(x, u)$, compute $u : [0, T] \to \mathbb{R}^k$ to minimize $\quad \psi(x(T)) + \int_0^T \phi(x, u)dt$

Pontryagin Minimum Principle: $\quad u = \mathcal{FBS}[u]$



$\mathcal{F}: \quad \dot{x} = \mathsf{F}(x, u)$

$\mathcal{B}: \quad \dot{\lambda} = -\mathsf{J}_{\mathsf{F}}^{\top}(x, u)\lambda - \phi_x(x, u)$

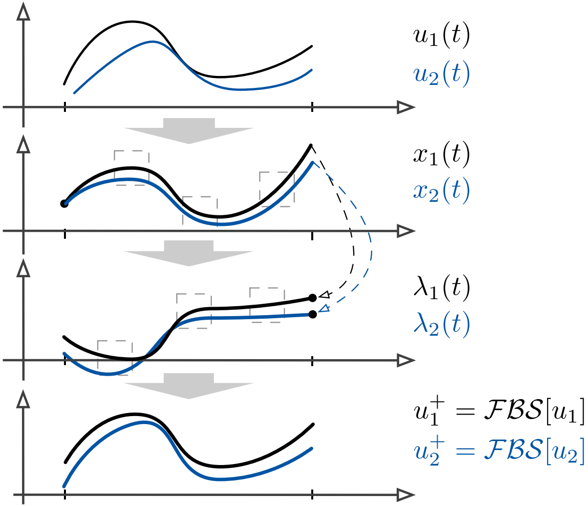
$\mathcal{S}: \quad u = \mathrm{argmin}_{\tilde{u}} \underbrace{\lambda^{\top}\mathsf{F}(x, \tilde{u}) + \phi(x, \tilde{u})}_{\mathcal{H}(x, \tilde{u}, \lambda)}$

To compute a solution to:

$$u = \mathcal{FBS}[u]$$

adopt

$$u^+ = \mathcal{FBS}[u]$$



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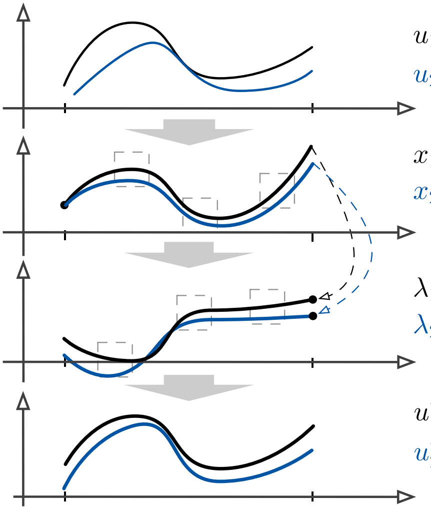

If $\text{osLip}_x(\text{F}) = -c$ and all other maps are Lipschitz,

1. $\text{osLip}_\lambda(\text{Adjoint}(\text{F})) \;=\; \text{osLip}_x(\text{F})$
2. $\text{Lip}(\mathcal{FBS}) = \text{constant} \times \dfrac{1 - \mathrm{e}^{-cT}}{c}$

$\mathcal{FBS}$ contracting for short T or large c

Summary

contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

From closed to open, interconnected and optimal systems:

1. iISS
2. network small gain theorems
3. numerical optimal control

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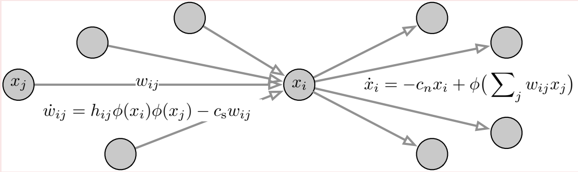
Applications coupled neural-synaptic dynamics and ML via optimal control



Outline

From nominal to uncertain systems

Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x) + \Delta(x)$$

Assume:

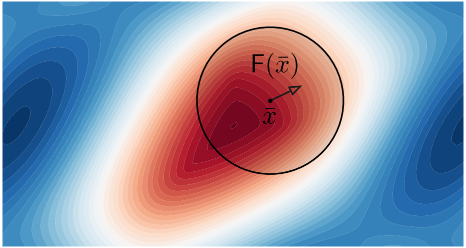
- **contractivity**: $\text{osLip}(\mathsf{F}) \leq -c < 0$
- **bounded disturbance**: $\text{osLip}(\Delta) \leq d < c$

Then

1. $\mathsf{F} + \Delta$ is strongly contracting with rate $c - d$
2. the unique equilibria x^*_{F} of F and $x^*_{\mathsf{F}+\Delta}$ of $\mathsf{F} + \Delta$ satisfy

$$\|x^*_{\mathsf{F}} - x^*_{\mathsf{F}+\Delta}\| \leq \frac{\|\Delta(x^*_{\mathsf{F}})\|}{c - d}$$

From global to local contractivity



Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x)$$

Assume:

- **contractivity over closed set D**: $\text{osLip}(\mathsf{F}|_D) \leq -c < 0$
- **existence of almost equilibrium**: D contains the closed B at $\bar{x}$ of radius $r \geq \|\mathsf{F}(\bar{x})\|/c$

Then

1. B is forward invariant
2. $\mathsf{F}|_B$ is strongly infinitesimally contracting

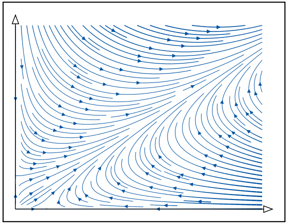
From strongly to weakly contracting systems

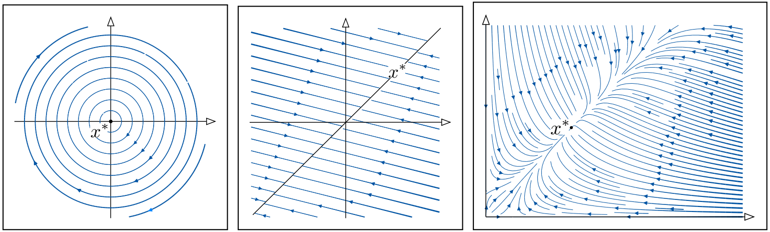
Given a norm $\|\cdot\|$, consider

$$\dot{x} = \mathsf{F}(x) \qquad \text{satisfying} \qquad \mathsf{osLip}(\mathsf{F}) = 0$$

Dichotomy for weakly-contracting systems

1. no equilibrium and every trajectory is unbounded, or
2. at least one equilibrium, every trajectory is bounded, and local asy stability $\implies$ global





Outline

1. Contractivity of dynamical systems
   - From discrete-time to continuous-time dynamics
   - Table of infinitesimal contractivity conditions
   - Application to recurrent neural networks
   - Connection with convex optimization
2. From closed to open, interconnected and optimal systems
   - Incremental input-to-state stability
   - Interconnected contracting systems
   - Contractivity in indirect optimal control
3. Additional robustness, computational and stability properties
4. Conclusions and Future Research

Robust and computationally-friendly stability theory

1. contractivity conditions on normed vector spaces
2. convexity and fixed point methods
3. disturbances, interconnections and optimal control

	Lyapunov Theory	**Contraction Theory for Dynamical Systems**
	F admits global Lyapunov function	F is strongly contracting
existence of equilibrium	assumed	implied + computational methods
Lyapunov function	arbitrary	distance to trajectory (+ norm of vector field)
inputs	ISS via $\mathcal{KL}$ and $\mathcal{L}$ functions	iISS via explicit formulas

search for contraction properties
design engineering systems to be contracting

References (1/2)

Contraction theory on normed spaces:

- A. Davydov, S. Jafarpour, and F. Bullo. Non-Euclidean contraction theory for robust nonlinear stability. *IEEE Transactions on Automatic Control*, 67(12):6667–6681, 2022b.
- S. Jafarpour, A. Davydov, and F. Bullo. Non-Euclidean contraction theory for monotone and positive systems. *IEEE Transactions on Automatic Control*, 2023.

Contractivity in optimal control:

- K. D. Smith and F. Bullo. Contractivity of the method of successive approximations for optimal control. *IEEE Control Systems Letters*, Nov. 2022.

Contracting neural networks and fixed point theory:

- S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021.
- A. Davydov, A. V. Proskurnikov, and F. Bullo. Non-Euclidean contractivity of recurrent neural networks. In *American Control Conference*, pages 1527–1534, Atlanta, USA, May 2022d.
- F. Bullo, P. Cisneros-Velarde, A. Davydov, and S. Jafarpour. From contraction theory to fixed point algorithms on Riemannian and non-Euclidean spaces. In *IEEE Conf. on Decision and Control*, Dec. 2021.

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Here at CDC 2022

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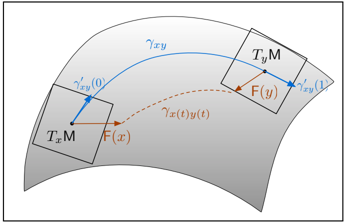
- A. Davydov, S. Jafarpour, M. Abate, F. Bullo, and S. Coogan. Comparative analysis of interval reachability for robust implicit and feedforward neural networks. In *IEEE Conf. on Decision and Control*, 2022a. URL **https://arxiv.org/abs/2204.00187**
- V. Centorrino, F. Bullo, and G. Russo. Contraction analysis of Hopfield neural networks with Hebbian learning. In *IEEE Conf. on Decision and Control*, Dec. 2022.
- R. Ofir, F. Bullo, and M. Margaliot. Minimum effort decentralized control design for contracting network systems. *IEEE Control Systems Letters*, 6:2731–2736, 2022.

Resources on contraction theory for dynamics, control and learning

1. tutorial session "Contraction Theory for Machine Learning" at the 2021 IEEE CDC conference:
   https://sites.google.com/view/contractiontheory
2. free online book and 10h minicourse
   http://motion.me.ucsb.edu/book-ctds
   https://youtu.be/RvR47ZbqJjc
3. upcoming Workshop on "Contraction Theory for Systems, Control, and Learning" at the 2023 American Control Conference in San Diego, California (under review):
   http://motion.me.ucsb.edu/contraction-workshop-2023

Theoretical frontiers

- higher order contraction
- relationship with monotone operator theory
- metric spaces: seminorms, Hilbert metrices ...



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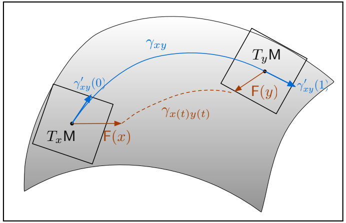
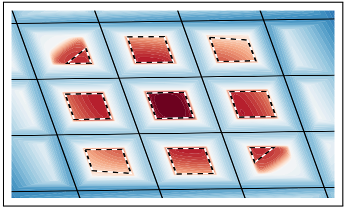


Limitations: not all stable systems are contractive:

- Lyapunov-diagonally-stable networks
- multistable systems
- biochemical networks



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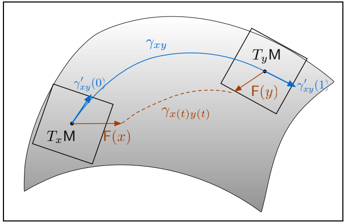

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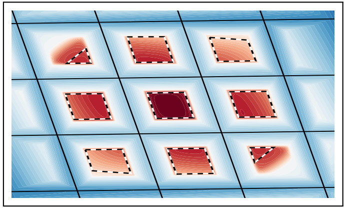

Application to control and learning

1. control: optimization-based control design
2. ML: implicit models and energy-based learning
3. neuroscience: robust dynamical modeling