NONLINEAR SYSTEMS with LIMITED DATA: ESTIMATION, CONTROL and SYNCHRONIZATION

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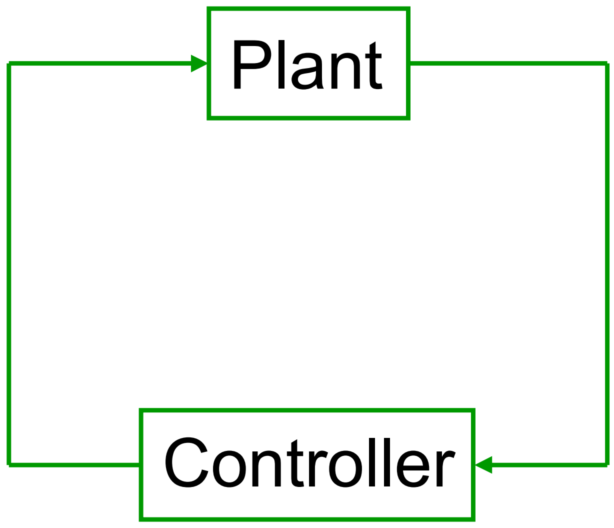
INFORMATION FLOW in CONTROL SYSTEMS



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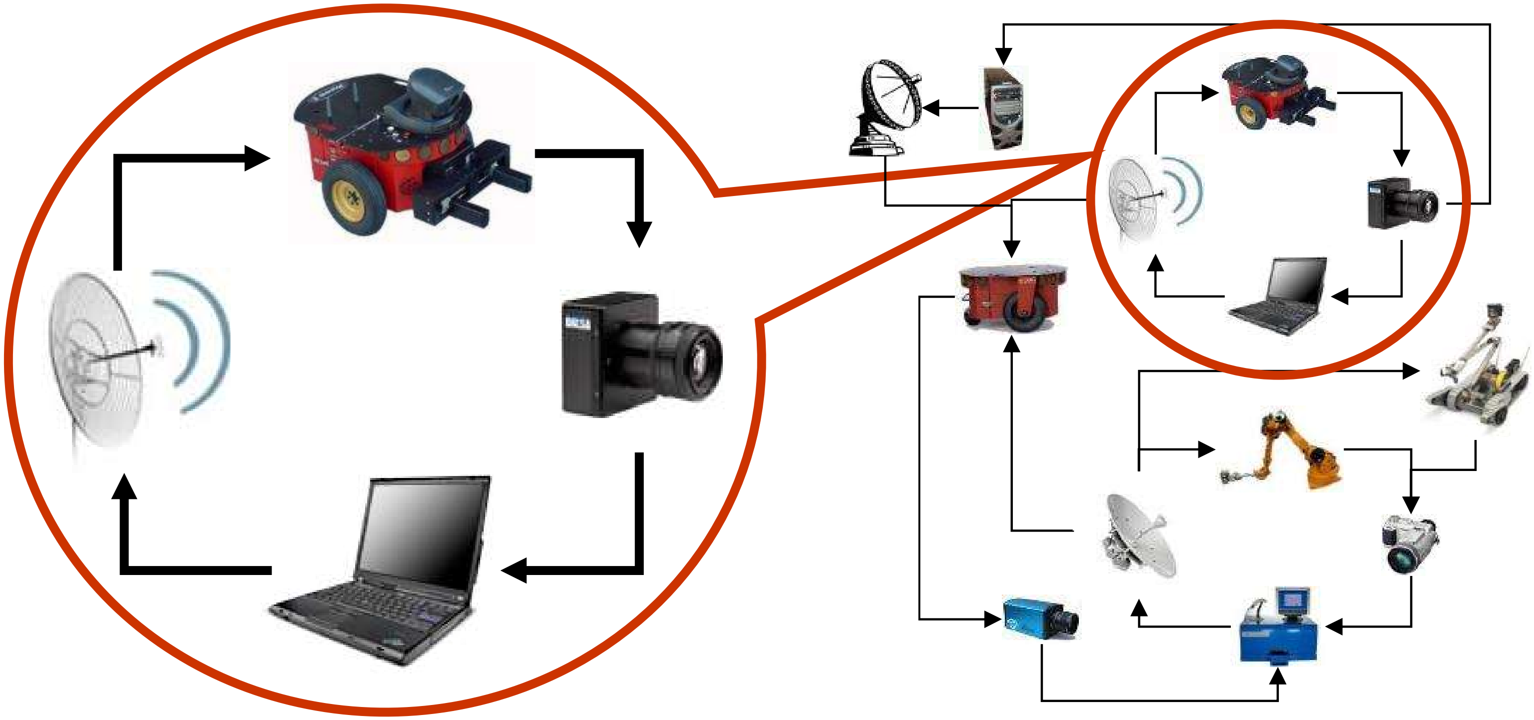


Limited channel capacity, data encryption, coarse sensing & actuation

$\Downarrow$

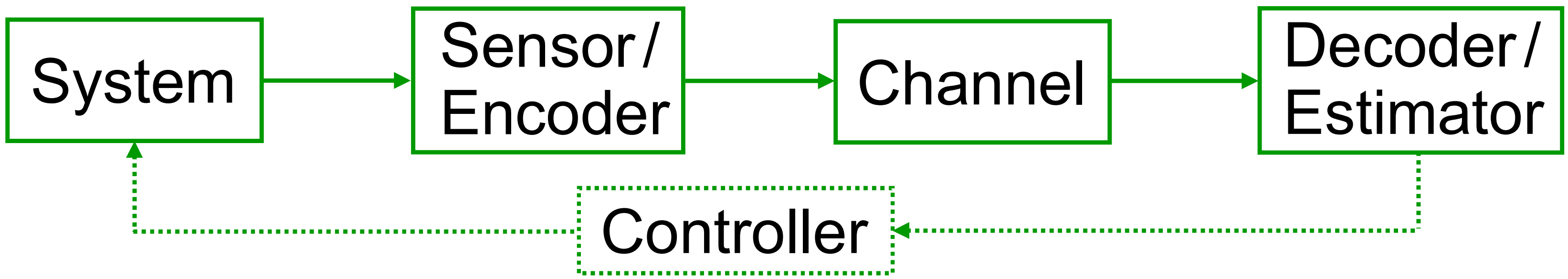
errors in signal measurement, transmission, and reconstruction

$\Downarrow$

need robust algorithms

TWO SPECIFIC SCENARIOS

- State estimation and model detection with finite data rate: an entropy approach



- Observers robust to measurement errors, with applications to control and synchronization

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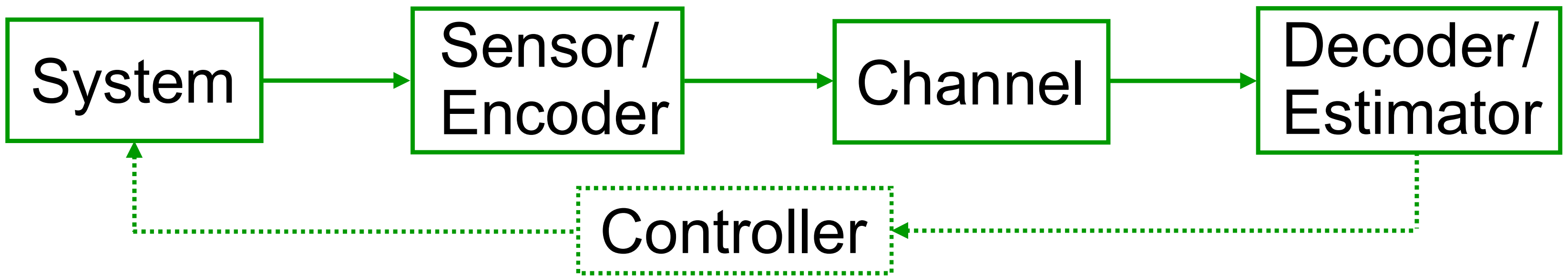
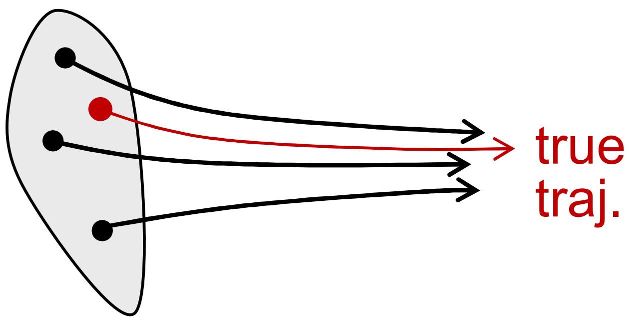


- Observers robust to measurement errors, with applications to control and synchronization

BASIC MOTIVATING QUESTION

How much data is needed to estimate the system's state?

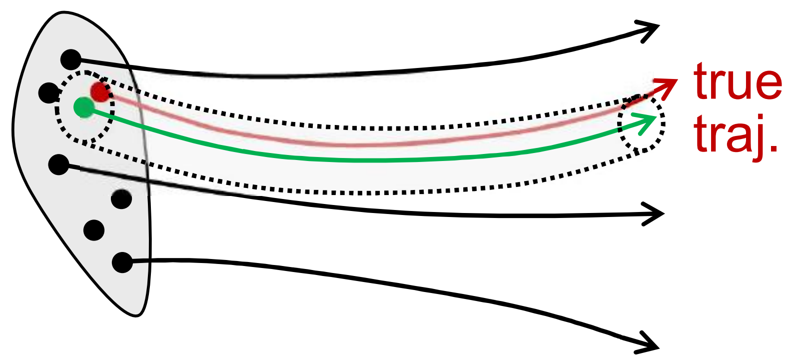
Contractive system:



Any trajectory can be used to approximate the real one

$\Longrightarrow$ no data needed

General system:



How many trajectories (or initial states) are needed to approximate all others?

need to make this precise

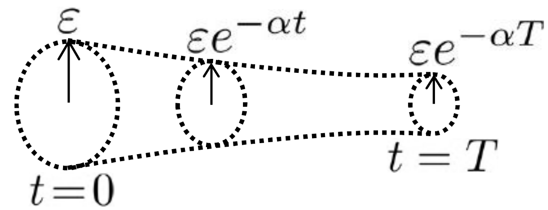
This relates to entropy and data rate

AN ENTROPY NOTION

$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad x(0) \in K$ – known compact set

$\xi(x,t)$ – solution from initial state x after time t

Pick: time horizon $T > 0$, resolution $\varepsilon > 0$,
desired exponential convergence rate[1] $\alpha \geq 0$



A set of points $x_1, \ldots, x_N \in K$ is (T, ε)-spanning if $\forall x \in K \ \exists x_i$:

$$|\xi(x,t) - \xi(x_i,t)| < \varepsilon e^{-\alpha t} \quad \forall t \in [0,T]$$

$s(T, \varepsilon) :=$ cardinality N of smallest (T, ε)-spanning set

Estimation entropy:

$$\boxed{h(f) := \lim_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{1}{T} \log s(T, \varepsilon)}$$

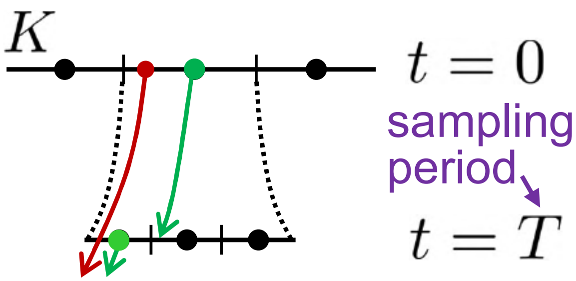
Kolmogorov, Sinai, Adler, …, Boichenko, Colonius, Kawan, Leonov, Matveev, Nair, Pogromsky, Savkin, …

[1] L, Mitra, Entropy and minimal bit rates for state estimation and model detection, TAC, 2018

TOY EXAMPLE

$\dot{x} = \lambda x, \quad \lambda > 0, \; x(0) \in K \subset \mathbb{R}$ – known compact interval

Goal: estimate $x(t)$ using finite-data-rate encoding of x-values



- divide K into N equal intervals with centers x_i
- record the index of the interval containing $x(0)$
- divide reachable set again into N subintervals
- repeat

This encoding scheme uses data at $\frac{1}{T}\log N$ bits per time unit

At $t = \ell T$, we know $x(t)$ is in an interval of length $\frac{|K|}{N^\ell} e^{\ell \lambda T}$

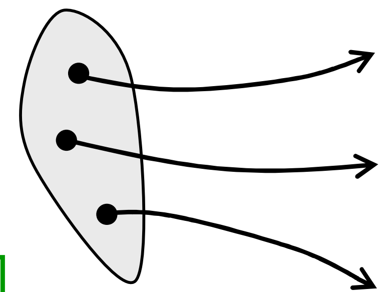
To estimate $x(t)$ with error converging to 0 as $e^{-\alpha t}$ we need
$N \geq e^{(\lambda+\alpha)T} \implies$ need data rate of $\boxed{\lambda + \alpha \text{ bits}}$ (or nats)

Entropy: the set $C := \{x_1, \ldots, x_N\}$ is (T, ε)-spanning if
$|x_i - x_{i+1}| < \varepsilon e^{-(\lambda+\alpha)T} \Rightarrow \#C = e^{(\lambda+\alpha)T}|K|/\varepsilon$
$\limsup_{T\to\infty} \frac{1}{T}\log$ of this gives $\boxed{h = \lambda + \alpha}$

CONTRACTION / EXPANSION RATE

Back to general case: $\underline{x} = f(x)$; $x(0) \in K \subset \mathbb{R}^n$

»(x; t) – solution from x after time t



We want to find a constant $c \in \mathbb{R}$ s.t.

$$\boxed{|\xi(x_1, t) - \xi(x_2, t)| \leq e^{ct}|x_1 - x_2|}$$

as long as solutions stay in a compact set (or globally)

E.g., c can be Lipschitz constant of f

If f is C^1, a sharper bound is obtained with $c := \sup_x \mu\left(\frac{\partial f}{\partial x}(x)\right)$

where $\frac{\partial f}{\partial x}$ is Jacobian matrix and

$\mu(A) := \lim_{\varepsilon \searrow 0} \frac{\|I+\varepsilon A\|-1}{\varepsilon}$ is matrix measure

(e.g., for ∞-norm $\mu(A) = \max_i \{a_{ii} + \sum_{j \neq i} |a_{ij}|\}$)

BOUNDS on ENTROPY

$\dot{x} = f(x)$; $x(0) \in K \subset \mathbb{R}^n$, $|\xi(x_1,t) - \xi(x_2,t)| \le e^{ct}|x_1 - x_2|$

Upper bound: $\boxed{h(f) \le \max\{(c+\alpha)n, 0\}}$

Sketch of proof:

- centers of balls of radius $\varepsilon e^{-(c+\alpha)T}$ that cover K form a (T,ε)-spanning set $\Rightarrow$ need to count them
- if we use, e.g., ∞-norm balls (cubes), need $e^{(c+\alpha)T}/\varepsilon$ per dimension to cover a unit hypercube
- $\limsup\limits_{T\to\infty} \dfrac{1}{T} \log \left(e^{(c+\alpha)T}/\varepsilon\right)^n = (c+\alpha)n$ $\square$

BOUNDS on ENTROPY

$\dot{x} = f(x)$; $x(0) \in K \subset \mathbb{R}^n$, $|\xi(x_1,t) - \xi(x_2,t)| \leq e^{ct}|x_1 - x_2|$

Upper bound: $\boxed{h(f) \leq \max\{(c+\alpha)n, 0\}}$

For linear system $\dot{x} = Ax$ this result can be refined to

$$\boxed{h(A) = \sum_{i=1}^{n} \max\{\mathrm{Re}\,\lambda_i(A) + \alpha, 0\}}$$

Lower bound comes from computing $\mathrm{vol}(\xi(K,t))$ by Liouville's trace formula and counting # of balls that can cover this volume[1,2]

Similar argument[3] gives a lower bound for nonlinear system:

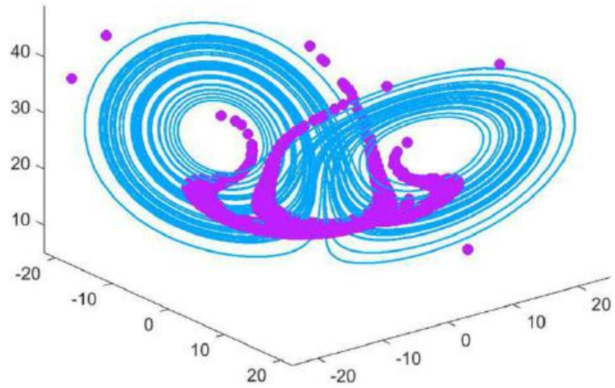
$$\boxed{h(f) \geq \inf_x \, \mathrm{tr}\, \frac{\partial f}{\partial x}(x) + \alpha n}$$

[1] Savkin, Analysis and synthesis of networked control systems, Automatica, 2006

[2] Schmidt, MS Thesis, UIUC, 2016

[3] Colonius, Minimal bit rates and entropy for exponential stabilization, SICON, 2012

EXAMPLE: LORENZ SYSTEM



EXAMPLE: LORENZ SYSTEM

$$\begin{aligned}\dot{x}_1 &= \sigma x_2 - \sigma x_1\\ \dot{x}_2 &= \theta x_1 - x_2 - x_1 x_3\\ \dot{x}_3 &= -\beta x_3 + x_1 x_2\end{aligned}$$

For initial set $K = B_{r_0}\big((0,0,0)\big)$

can compute r s.t. $x(t) \in B_r\big((0,0,\sigma+\theta)\big)\ \forall t \geq 0$

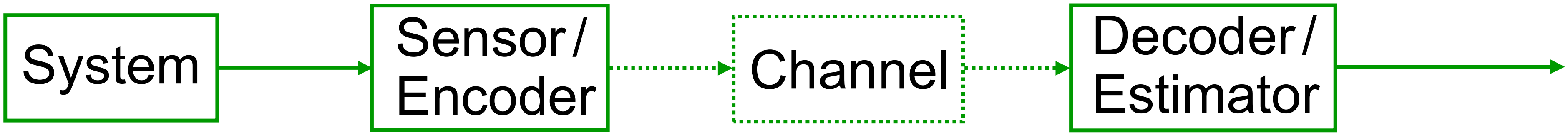
Jacobian is $J(x) := \dfrac{\partial f}{\partial x}(x) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \theta - x_3 & -1 & -x_1 \\ x_2 & x_1 & -\beta \end{pmatrix}$

Its matrix measure is $\mu(J(x)) = \max\limits_{i=1,2,3}\Big\{J_{ii}(x) + \sum\limits_{j\neq i} |J_{ij}(x)|\Big\}$

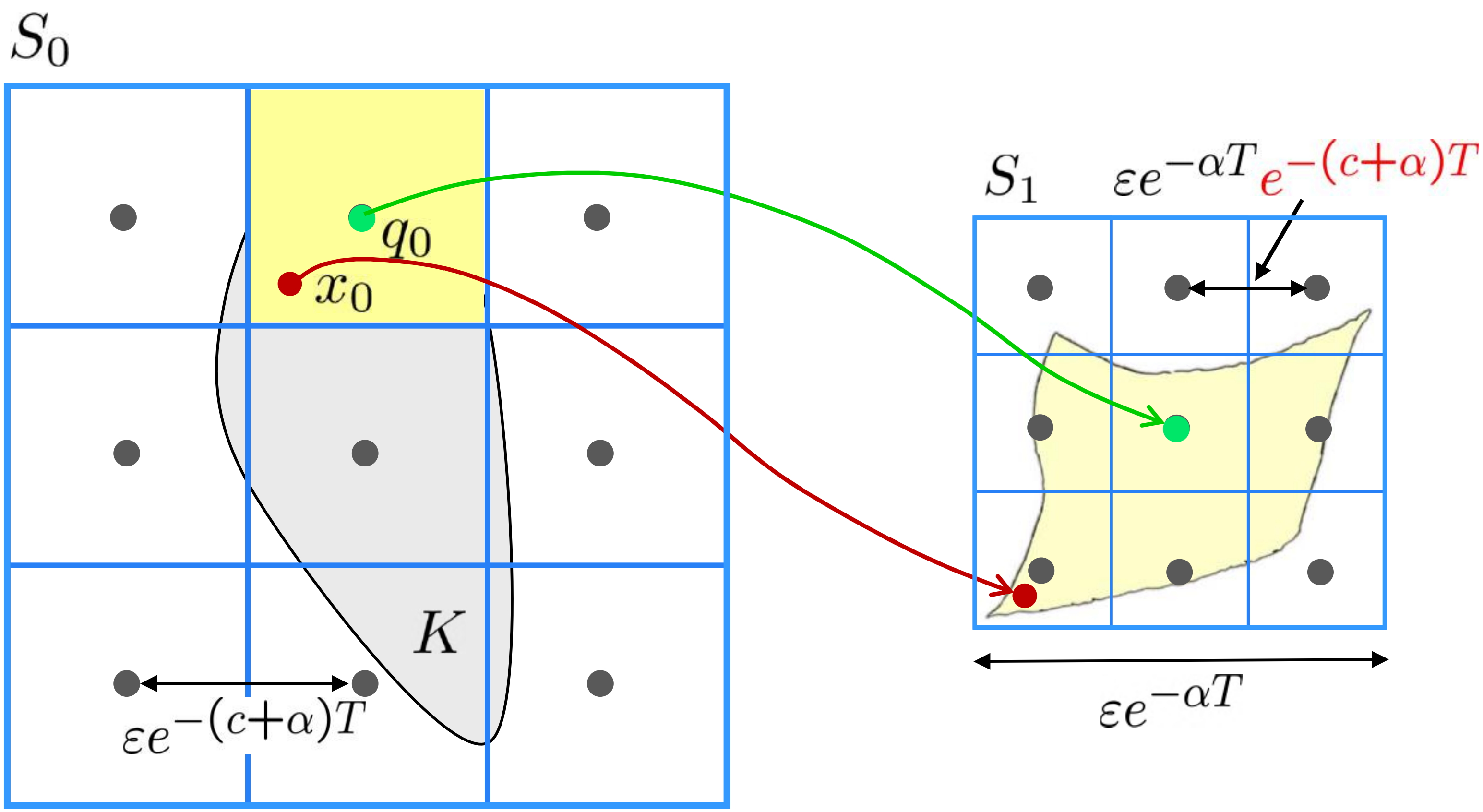
hence $c = \max\limits_{x\in B_r} \mu(J(x)) = \max\big\{0, -1+\sigma+2r, -\beta+2r\big\}$

and $h(f) \leq 3(c+\alpha)$

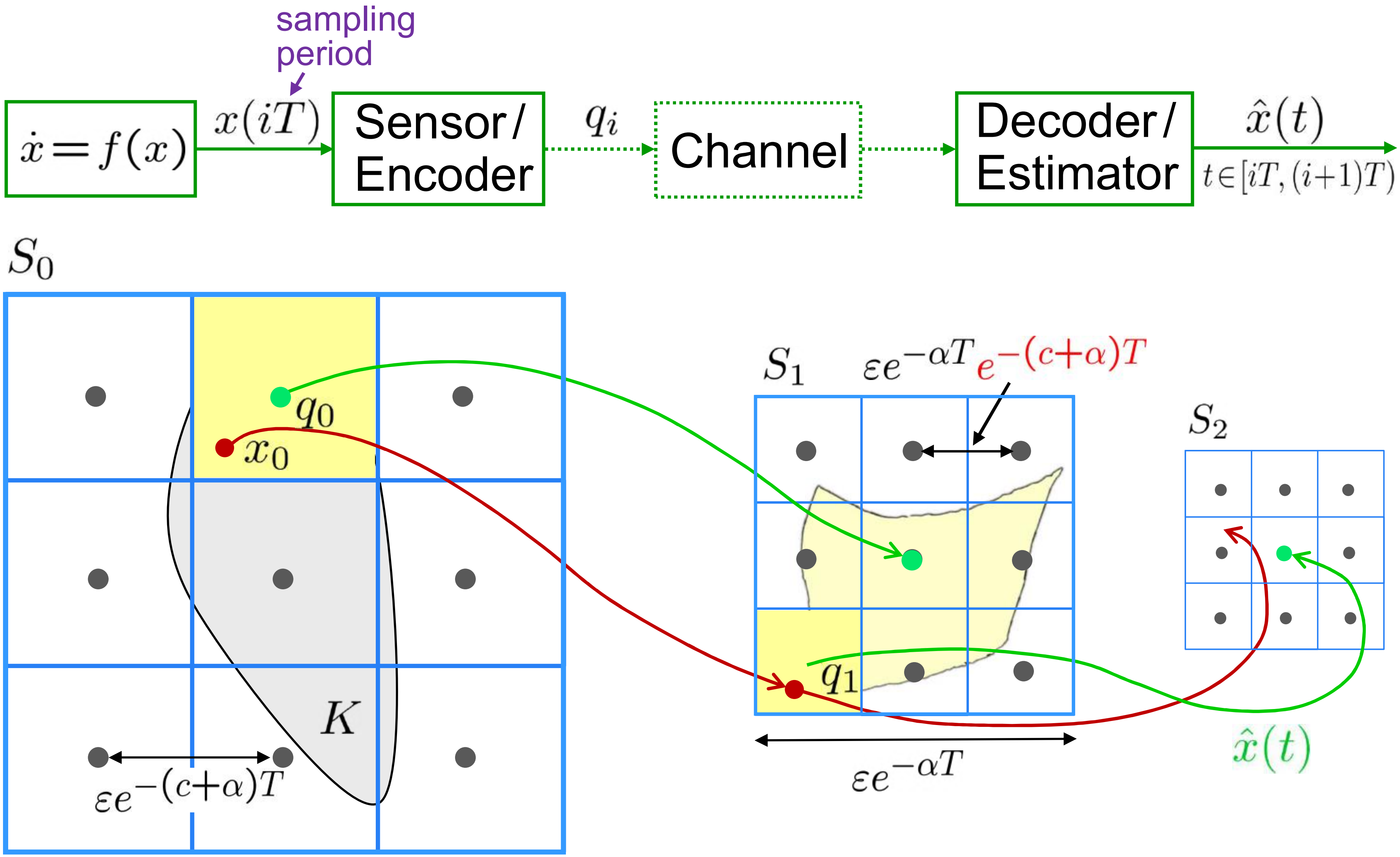
STATE ESTIMATION PROCEDURE



STATE ESTIMATION PROCEDURE



STATE ESTIMATION PROCEDURE



Properties: $x(iT) \in S_i \ \forall i$ and $\|x(t) - \hat{x}(t)\|_\infty \leq \varepsilon e^{-\alpha t} \ \forall t$

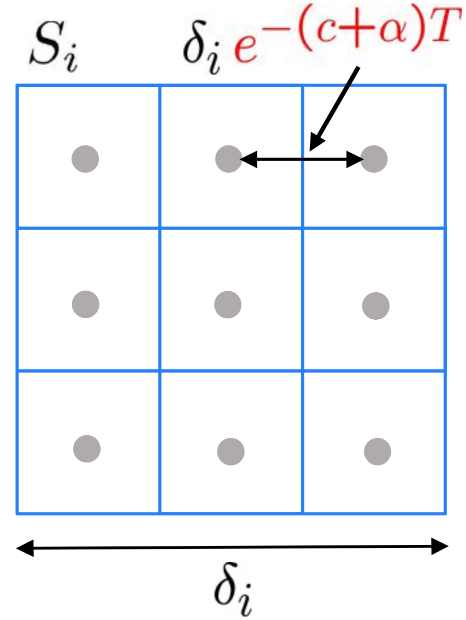
DATA RATE and EFFICIENCY GAP

$S_i, i \geq 1$ is divided into $e^{(c+\alpha)T}$ sub-boxes per dim

$\Rightarrow$ q_i is drawn from alphabet of size $N = e^{(c+\alpha)Tn}$

$\Rightarrow$ bit rate is $\frac{1}{T}\log N = (c+\alpha)n$

which is our upper bound on $h(f)$



In fact, quantization points define a spanning set

More precisely: # of possible codewords over ℓ rounds, N^ℓ, equals cardinality of $(\ell T, \varepsilon)$-spanning set

$$\Rightarrow \text{ bit rate } = \frac{1}{\ell T}\log N^\ell \geq \frac{1}{\ell T}\log s(\ell T, \varepsilon) \xrightarrow[\ell\to\infty,\ \varepsilon\to 0]{} h(f)$$

smallest spanning set

So, entropy gives the minimal required data rate for state estimation[1]

Efficiency gap of our algorithm is $(c+\alpha)n - h(f)$

which is the price to pay for having a constructive procedure

[1] Savkin, Analysis and synthesis of networked control systems, Automatica, 2006

MODEL DETECTION PROBLEM

Want to distinguish between two competing system models

$$\dot{x} = f_1(x), \qquad \dot{x} = f_2(x)$$

using finite-data-rate state measurements (as before)

Need the two systems to be "sufficiently different"

$\xi_i(x,t)$ – solution of system i from initial state x after time t ,

T – sampling period, c_1 – expansion rate of system 1

Call the two models separated if $\exists\, \varepsilon^* > 0$ s.t. $\forall\, \varepsilon \le \varepsilon^*$:

$$|x_1 - x_2| \le \varepsilon \;\Rightarrow\; |\xi_1(x_1,T) - \xi_2(x_2,T)| > \varepsilon e^{c_1 T}$$

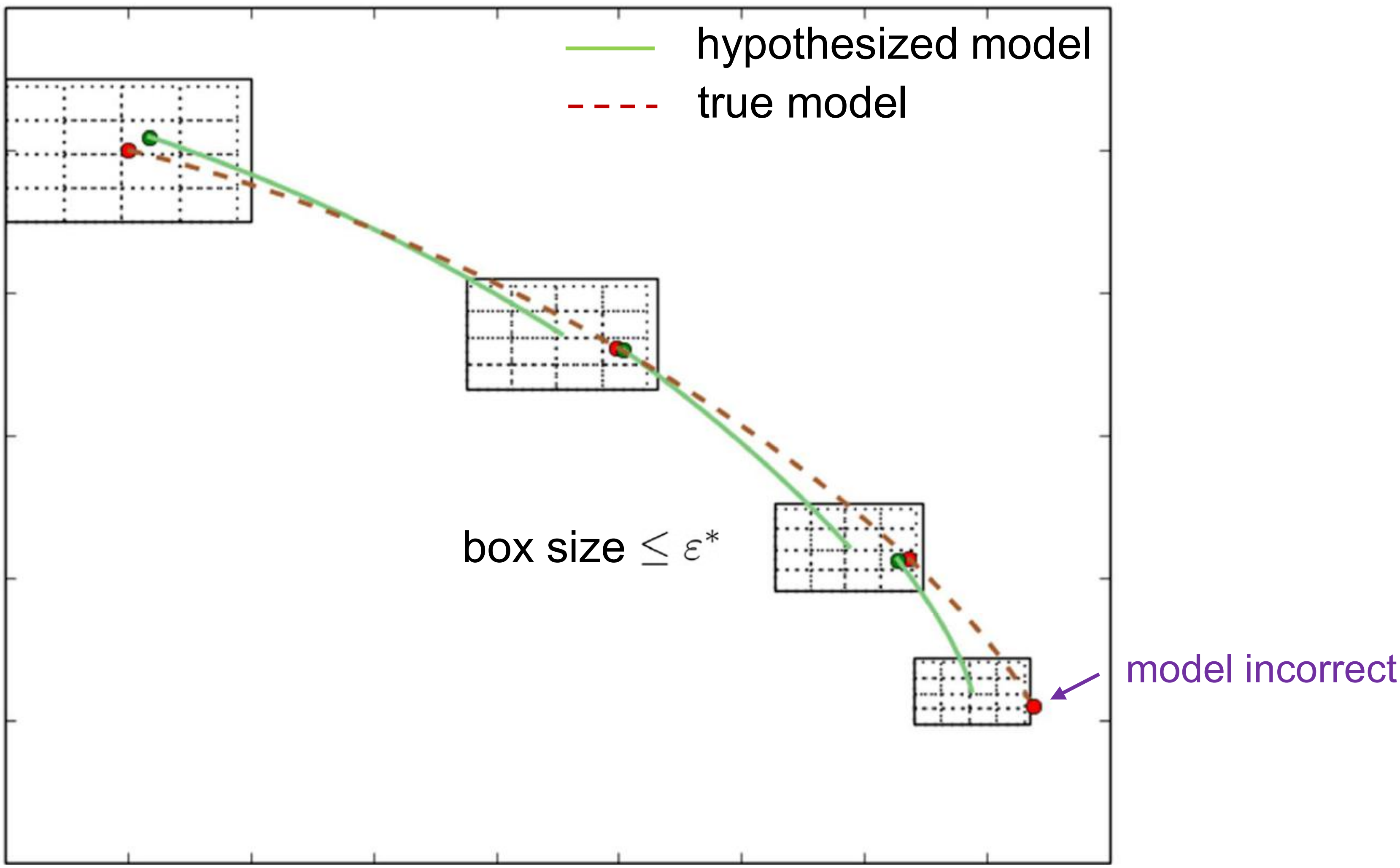
Interpretation: for nearby initial states, trajectories of the two systems diverge faster than would be possible if they both came from system 1

Separation property holds in generic situations, if T is small enough[1]

[1] L, Mitra, Entropy and minimal bit rates for state estimation and model detection, TAC, 2018

MODEL DETECTION ALGORITHM

With separation assumption, our previous state estimation algorithm will eventually falsify model 1 if it is incorrect

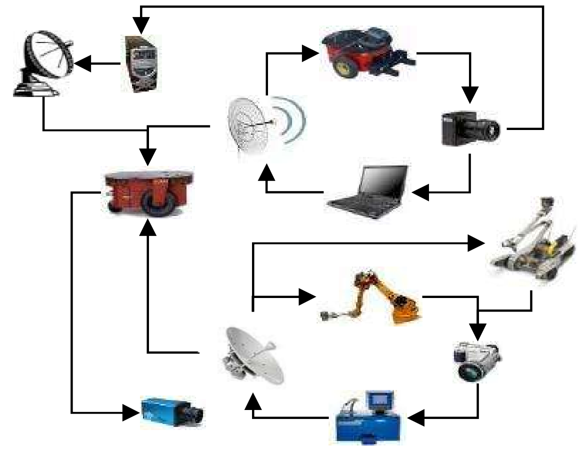


With prior knowledge of ε^*, it will also certify model 1 if it is correct

ONGOING WORK: INTERCONNECTED SYSTEMS

$$\boxed{\dot{x}_i = f_i(x_1, \dots, x_k), \; i = 1, \dots, k}$$

$$\dim(x_i) = n_i, \; n_1 + \cdots + n_k = n$$



Jacobian blocks: $J_{ij}(x) = (\partial f_i / \partial x_j)(x)$

Assume: $\mu(J_{ii}(x)) \le a_{ii}, \; \|J_{ij}(x)\| \le a_{ij} \; \forall\, x, \; \forall\, i, j$

Structure matrix: $A := (a_{ij})_{i,j=1}^{k}$

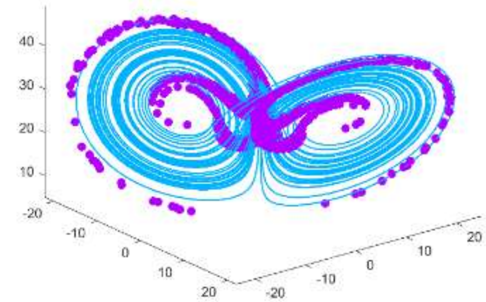
A is a Metzler matrix $\Rightarrow$ eigenvalue $\lambda_{\max}(A)$ is real

Entropy bound[1]: $\boxed{h(f) \le \max\{(\lambda_{\max}(A) + \alpha)n, 0\}}$

[1] L, On topological entropy of interconnected nonlinear systems, IEEE CSL/CDC, 2021

EXAMPLE: LORENZ SYSTEM (revisited)

$\underline{x}_1$ = ¾x$_2$ ¡ ¾x$_1$
$\underline{x}_2$ = µx$_1$ ¡ x$_2$ ¡ x$_1$x$_3$
$\underline{x}_3$ = ¡ x$_3$ + x$_1$x$_2$



Can view the system as interconnection of 3 scalar subsystems

Jacobian is $J(x) = \begin{pmatrix} -\sigma & \sigma & 0 \\ \theta - x_3 & -1 & -x_1 \\ x_2 & x_1 & -\beta \end{pmatrix}$

For $K = B_{r_0}\big((0,0,0)\big)$ we have $x(t) \in B_r\big((0,0,\sigma+\theta)\big)\ \forall t \geq 0$

Need matrix $A = (a_{ij})$ s.t. $\forall x \in B_r$: $\mu(J_{ii}(x)) \leq a_{ii}$, $\|J_{ij}(x)\| \leq a_{ij}$

Can take $A = \begin{pmatrix} -\sigma & \sigma & 0 \\ \sigma + r & -1 & r \\ r & r & -\beta \end{pmatrix}$

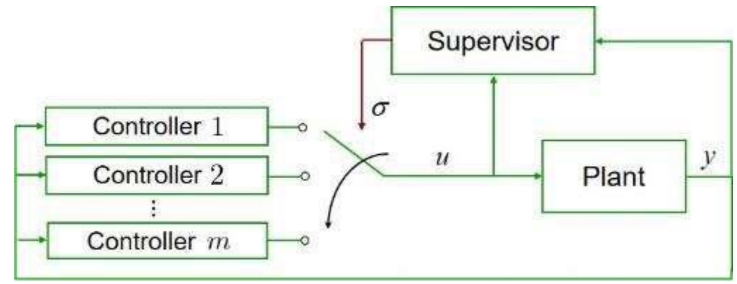
Previous result gives
$h(f) \leq 3(\lambda_{\max}(A) + \alpha)$

Improves on earlier matrix measure bound, but far from being tight[1]

[1] Pogromsky, Matveev, Estimation of topological entropy via direct Lyapunov method, Nonlinearity, 2011

ONGOING WORK: SWITCHED SYSTEMS

$$\boxed{\dot{x} = f_\sigma(x)}$$



- $\dot{x} = f_p(x); \ p \in P$ are modes
- $\sigma : [0, \infty) \to \mathcal{P}$ is a switching signal

Can define entropy as before for each fixed switching signal

For each mode p, define active time $\tau_p(t) := \int_0^t \mathbf{1}_p(\sigma(s))ds$ and active rate $\rho_p(t) := \tau_p(t)/t$ – these play a role in entropy bounds

For example, entropy of switched linear system $\dot{x} = A_\sigma x$ satisfies[1]

$$\limsup_{t\to\infty} \sum_p \operatorname{tr}(A_p)\rho_p(t) \le h(A_\sigma) \le \limsup_{t\to\infty} \sum_p n\mu(A_p)\rho_p(t)$$

Extensions to switched nonlinear systems also possible[2]

These bounds can inform control design for switched systems

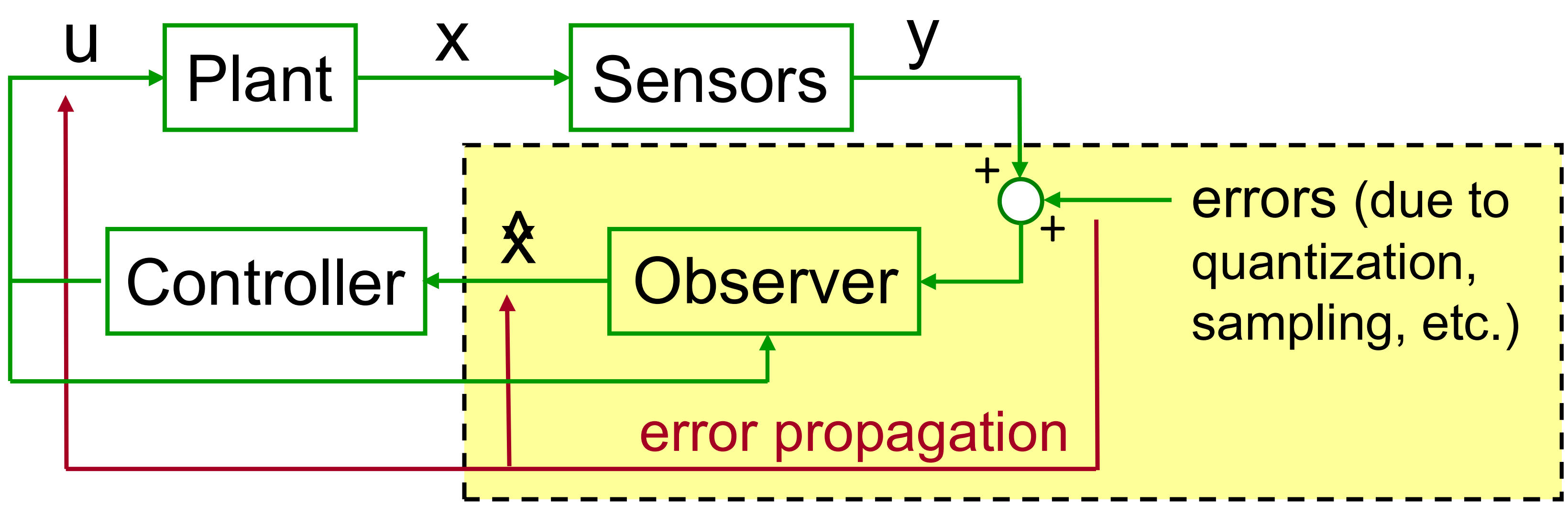
[1] Yang, Schmidt, L, Hespanha, Topological entropy of switched linear systems, MCSS, 2020; $\alpha = 0$

[2] Yang, L, Hespanha, Topological entropy of switched nonlinear systems, HSCC, 2021

TWO SPECIFIC SCENARIOS

- State estimation and model detection with finite data rate: an entropy approach

- Observers robust to measurement errors, with applications to control and synchronization



Few results for nonlinear systems are available[1,2]

[1] Khalil, Praly, High-gain observers in nonlinear feedback control, IJRNC, 2013

[2] Chong, Postoyan, Nesic, Kuhlmann, Varsavsky, A robust circle criterion observer, Automatica, 2012

SENSITIVITY vs. ROBUSTNESS

$\boxed{\dot{x} = f(x, d)}$ $\quad x$ – state, $\quad d$ – disturbance

Asymptotic stability for $d \equiv 0$ does not imply bounded response to bounded disturbances:

$$\dot{x} = -x + xd \qquad (x \text{ unbounded for } d \equiv 2)$$

or converging response to vanishing disturbances:

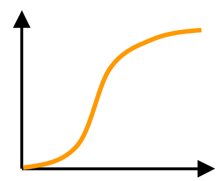
$$\dot{x} = -x + x^2 d \qquad (\text{may have } x \uparrow \infty \text{ even if } d \to 0)$$

Both properties are captured by input-to-state stability (ISS)[1]:

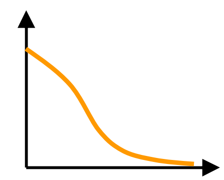
$$|x(t)| \le \underbrace{\beta}_{\in\, \mathcal{KL}}(|x(0)|, t) + \underbrace{\gamma}_{\in\, \mathcal{K}}\big(\|d\|_{[0,t]}\big)$$

sup norm

("ISS gain")



class $\mathcal{K}$



class $\mathcal{L}$

This will be our benchmark robustness notion, with some caveats

[1] Sontag, Smooth stabilization implies coprime factorization, TAC, 1989

ASYMPTOTIC–RATIO ISS LYAPUNOV FUNCTIONS[1]

These are functions $V(x)$ whose derivative along solutions satisfies

$$\dot{V} \le -\alpha(|x|) + g(|x|, |d|)$$

where $\alpha \in \mathcal{K}$, g is continuous non-negative, $g(r, \cdot) \in \mathcal{K}$, and

$$\boxed{\limsup_{r \to \infty} \frac{g(r,s)}{\alpha(r)} < 1 \quad \forall s \ge 0}$$

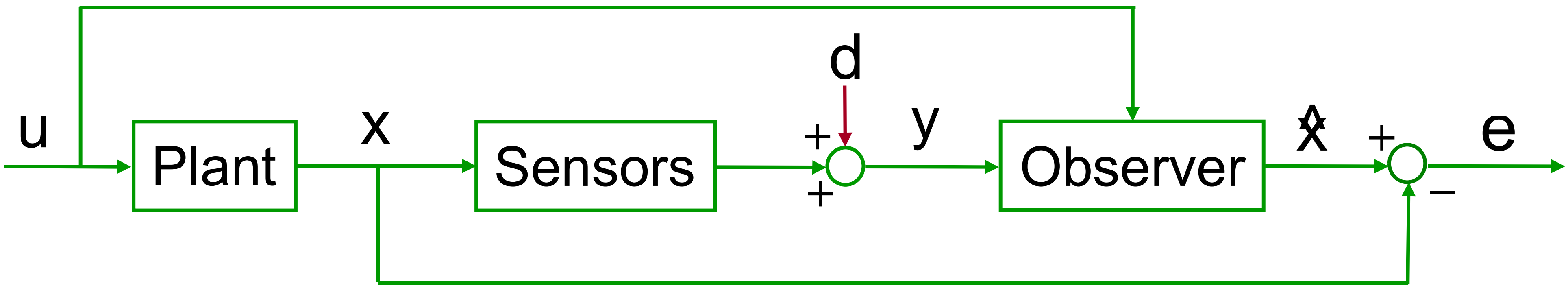
Can show: ISS $\Leftrightarrow$ $\exists$ asymptotic-ratio ISS Lyapunov function (by reducing to more standard Lyapunov characterizations of ISS)

Example (scalar): $\dot{x} = -\frac{1}{1+d^2}x + d, \quad V(x) := \frac{1}{2}x^2$

$$\dot{V} = -\frac{x^2}{1+d^2} + xd = \underbrace{-x^2}_{\alpha(|x|)} + \underbrace{x^2\frac{d^2}{1+d^2} + xd}_{g(|x|,|d|)} \quad \checkmark$$

[1] L, Shim, Asymptotic ratio characterization of input-to-state stability, TAC, 2015

OBSERVER SET-UP



Plant: $\dot{x} = f(x; u); \quad y = h(x; d) \quad (x \in \mathbb{R}^n)$

Observer: $\dot{z} = F(z; y; u); \quad \hat{x} = H(z; y) \quad (z \in \mathbb{R}^m)$

Full-order observer: $m = n,\ \hat{x} = z$; reduced-order: $m < n$

State estimation error: $e := \hat{x} - x$

Sensitivity issue[1]: can have $e \to 0$ when $d \equiv 0$

yet $e \to \infty$ for arbitrarily small $d \neq 0$

[1] Shim, Seo, Teel, Nonlinear observer design via passivation of error dynamics, Automatica, 2003

ROBUSTNESS of OBSERVER

Plant: $\dot{x} = f(x, u), \quad y = h(x, d)$

Observer: $\dot{z} = F(z, y, u), \quad \hat{x} = H(z, y)$

Estimation error: $e := \hat{x} - x$

ISS-like robustness: $\exists\, \beta \in \mathcal{KL},\ \gamma \in \mathcal{K}$ s.t.

$$|e(t)| \le \beta(|e(0)|, t) + \gamma\left(\|d\|_{[0,t]}\right)$$

Turns out to be too restrictive, not realistic

Modification: impose ISS only as long as x, u are bounded
(reasonable, as boundedness can come from controller design)

ROBUSTNESS of OBSERVER

Plant: $\dot{x} = f(x, u), \quad y = h(x, d)$

Observer: $\dot{z} = F(z, y, u), \quad \hat{x} = H(z, y)$

Estimation error: $e := \hat{x} - x$

ISS-like robustness: $\forall\, K > 0\ \exists\, \beta_K \in \mathcal{KL},\ \gamma_K \in \mathcal{K}$ s.t.

$$\boxed{|e(t)| \le \beta_K(|e(0)|, t) + \gamma_K(\|d\|_{[0,t]})}$$

whenever $\|u\|_{[0,t]}, \|x\|_{[0,t]} \le K$

Modification: impose ISS only as long as x, u are bounded
(reasonable, as boundedness can come from controller design)

Call such observers quasi-Disturbance-to-Error Stable (qDES)[1]

Accordingly, asymptotic-ratio Lyapunov condition only needs to hold for bounded x, u

[1] Shim, L, Nonlinear observers robust to measurement disturbances in an ISS sense, TAC, 2016

EXAMPLE: LINEARIZED ERROR DYNAMICS[1]

Plant: $\dot{x} = Ax + f(Cx; u); \quad y = Cx + d$

with $(A; C)$ detectable pair, so $\exists L$ s.t. $A - LC$ is Hurwitz

Observer: $\dot{z} = Az + f(y; u) + L(y - Cz); \quad \hat{x} = z$

Analysis of error dynamics: $e = z - x$

$V := e^\top P e$ where $P(A - LC) + (A - LC)^\top P = -I$

$$\dot{V} \le \overbrace{-|e|^2}^{\alpha(|e|)} + \overbrace{2|e|\|P\|\Big(\|L\||d| + \underbrace{\big|f(Cx + d, u) - f(Cx, u)\big|}_{\le \phi_K(|d|)}\Big)}^{g(|e|,|d|)\text{ – linear in } |e|}$$

Assume $|u|, |x| \le K$

Asymptotic ratio: $\frac{g(|e|,|d|)}{\alpha(|e|)} \xrightarrow[|e| \to \infty]{} 0 \;\Rightarrow$ observer is qDES

Also qDES are high-gain observer, circle-criterion observer

[1] Krener, Isidori, Linearization by output injection and nonlinear observers, SCL, 1983

REDUCED–ORDER qDES OBSERVERS

Plant (after a coordinate change):

$$\dot{x}_1 = f_1(x_1, x_2, u)$$

$$\dot{x}_2 = f_2(x_1, x_2, u)$$

$$y = x_1 + d$$

Observer:

$$\hat{x}_1 = y$$

$$\dot{z} = f_2(y, z, u)$$

$$\hat{x}_2 = z$$

$$e := z - x_2, \qquad V = V(e)$$

$$\dot{V} = \frac{\partial V}{\partial e}\Big(f_2(x_1, {\color{red}x_2 + e}, u) - f_2(x_1, {\color{green}x_2}, u)\Big)$$

Assume this is $\leq -\alpha(|e|)$, then we have an
asymptotic observer: $e \to 0$ (without d)

REDUCED–ORDER qDES OBSERVERS

Plant (after a coordinate change):

$$\dot{x}_1 = f_1(x_1, x_2, u)$$
$$\dot{x}_2 = f_2(x_1, x_2, u)$$
$$y = x_1 + d$$

Observer:

$$\hat{x}_1 = y$$
$$\dot{z} = f_2(y, z, u)$$
$$\hat{x}_2 = z$$

$$e := z - x_2, \qquad V = V(e)$$

$$\dot{V} = \frac{\partial V}{\partial e}\Big(f_2(y, x_2 + e, u) - f_2(x_1, x_2, u)\Big)$$

$$= \overbrace{\frac{\partial V}{\partial e}\Big[f_2(y, x_2+e, u) - f_2(y, x_2, u)\Big]}^{\text{assumed to be } \leq -\alpha(|e|)} + \frac{\partial V}{\partial e}\Big[f_2(y, x_2, u) - f_2(x_1, x_2, u)\Big]$$

REDUCED–ORDER qDES OBSERVERS

Plant (after a coordinate change):

$$\dot{x}_1 = f_1(x_1, x_2, u)$$
$$\dot{x}_2 = f_2(x_1, x_2, u)$$
$$y = x_1 + d$$

Observer:

$$\hat{x}_1 = y$$
$$\dot{z} = f_2(y, z, u)$$
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$$\dot{V} = \frac{\partial V}{\partial e}\left(f_2(y, x_2 + e, u) - f_2(x_1, x_2, u)\right)$$

$$\le \overbrace{\frac{\partial V}{\partial e}\Big[f_2(y, x_2 + e, u) - f_2(y, x_2, u)\Big]}^{\text{assumed to be } \le -\alpha(|e|)} + \overbrace{\left|\frac{\partial V}{\partial e}\right|}^{\le \rho(|e|)} \underbrace{\Big|f_2(y, x_2, u) - f_2(x_1, x_2, u)\Big|}_{\le \phi_K(|d|)}$$

Assume $|u|, |x| \le K$

We have: $\dot{V} \le -\alpha(|e|) + \rho(|e|)\phi_K(|d|)$

REDUCED–ORDER qDES OBSERVERS

Plant (after a coordinate change):

$$\dot{x}_1 = f_1(x_1; x_2; u)$$

$$\dot{x}_2 = f_2(x_1; x_2; u)$$

$$y = x_1 + d$$

Observer:

$$\hat{x}_1 = y$$

$$\dot{z} = f_2(y; z; u)$$

$$\hat{x}_2 = z$$

came from asymptotic observer property came from Lyapunov function

$$\boxed{\dot{V} \le -\alpha(|e|) + \rho(|e|)\phi_K(|d|)}$$

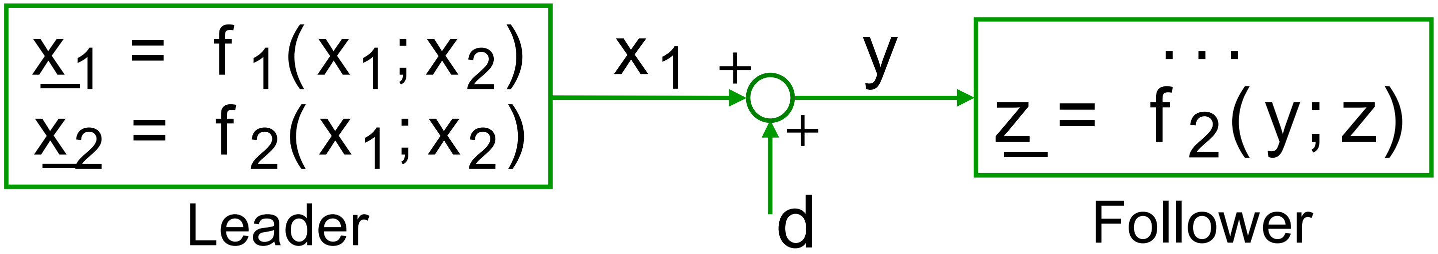
Asymptotic ratio condition:

$$\limsup_{r\to\infty} \frac{\rho(r)}{\alpha(r)} \phi_K(s) < 1 \ \forall s \quad \Leftarrow \quad \boxed{\lim_{r\to\infty} \frac{\rho(r)}{\alpha(r)} = 0}$$

Under this condition the observer is qDES

Synchronization examples that follow are analyzed in this way

ROBUST SYNCHRONIZATION and qDES OBSERVERS



$e := z - x_2$

Robust synchronization: $\forall K > 0 \ \exists \ \beta_K \in \mathcal{KL}, \gamma_K \in \mathcal{K}_\infty$ s.t.

$$\boxed{|e(t)| \le \beta_K(|e(0)|, t) + \gamma_K(\|d\|_{[0,t]})}$$

whenever $\|x\|_{[0,t]} \le K$ (in closed loop)

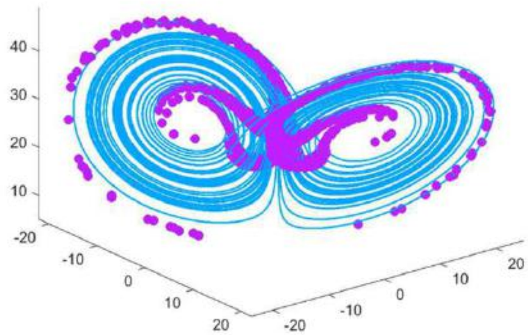
Equivalently: follower is a reduced-order qDES observer for leader

Sufficient condition from before: $\exists V = V(e)$ s.t. $\left|\frac{\partial V}{\partial e}\right| \le \rho(|e|)$,

$\frac{\partial V}{\partial e}(e)\Big(f_2(x_1, z) - f_2(x_1, x_2)\Big) \le -\alpha(|e|)$, and

$\limsup_{r \to \infty} \frac{\rho(r)}{\alpha(r)} = 0$ (asymptotic ratio condition)

APPLICATION EXAMPLE #1



Lorenz system

APPLICATION EXAMPLE #1

$$\begin{aligned}\dot{x}_1 &= \sigma x_2 - \sigma x_1\\ \dot{x}_2 &= \rho x_1 - x_2 - x_1 x_3\\ \dot{x}_3 &= - \quad x_3 + x_1 x_2\end{aligned}$$

Lorenz system

x_1 + d + → y

$$\begin{aligned}\dot{z}_2 &= \rho y - z_2 - y z_3\\ \dot{z}_3 &= - \quad z_3 + y z_2\end{aligned}$$

Observer

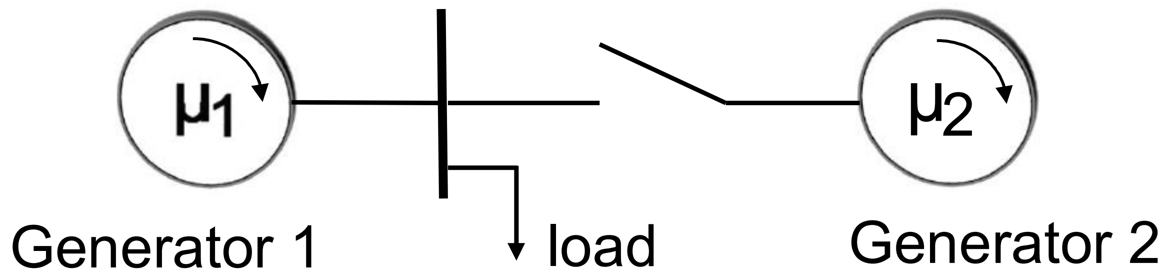
We already mentioned that x is bounded

Can show qDES from d to $e := \begin{matrix} z_2 - x_2 \\ z_3 - x_3 \end{matrix}$ using $V(e) = |e|^2$

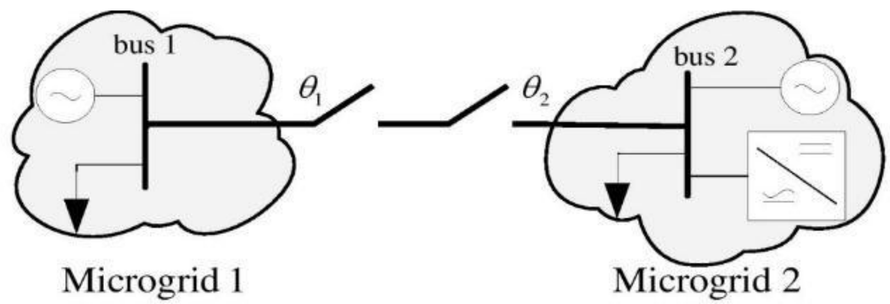
For d arising from time sampling and quantization, we can derive an explicit bound on synchronization error which is inversely proportional to data rate[1]

[1] Andrievsky, Fradkov, L, Robust Pecora-Carroll synchronization under communication constraints, SCL, 2018

APPLICATION EXAMPLE #2



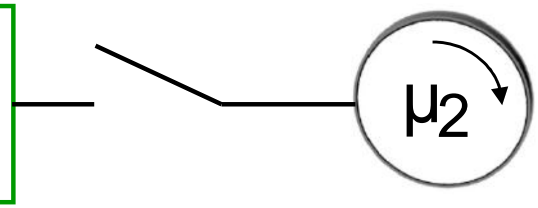
Baby version of microgrid synchronization



APPLICATION EXAMPLE #2

$$\dot{\theta}_1 = \omega_1$$
$$\dot{\omega}_1 = u_1 - \ell(t) - D_1\omega_1$$

Generator 1



Generator 2

$\ell(t)$ = electrical load (slowly varying)

$u_1(t; \theta_1)$ = control input (mechanical power)

With integral control: $\omega_1 \to$ desired freq. ω_0

APPLICATION EXAMPLE #2

$$\dot{\theta}_1 = \omega_1$$
$$\dot{\omega}_1 = u_1 - \ell(t) - D_1\omega_1$$

Generator 1

$$\dot{\theta}_2 = \omega_2$$
$$\dot{\omega}_2 = u_2 - D_2\omega_2$$

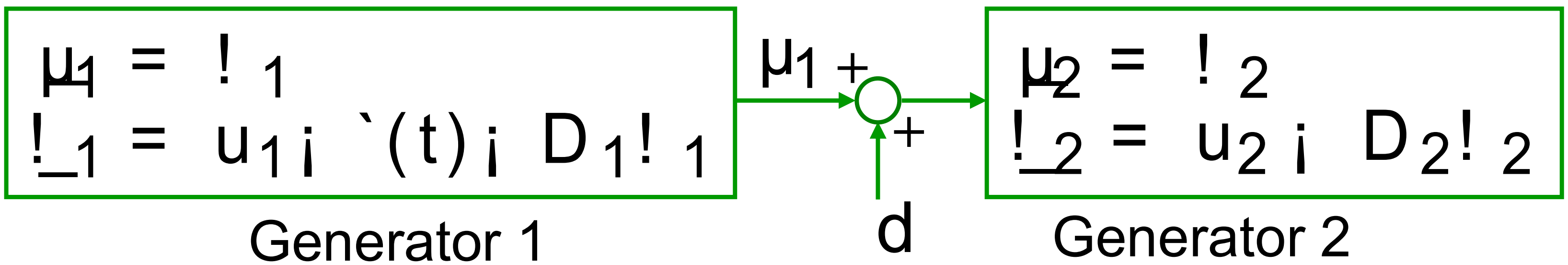
Generator 2

$\ell(t) =$ electrical load (slowly varying)

$u_1(t;\theta_1) =$ control input (mechanical power)

With integral control: $\omega_1 \to$ desired freq. ω_0

APPLICATION EXAMPLE #2



$\ell(t) =$ electrical load (slowly varying)

$u_1(t;\theta_1) =$ control input (mechanical power)

With integral control: $\omega_1 \to$ desired freq. ω_0

Measurements:
PMU corrupted
by disturbance

Objective: connect 2nd generator when $\theta_1 \approx \theta_2;\ \omega_1 \approx \omega_2$

- $V = e^2$ gives DES (ISS) from d to $e := \omega_2 - \omega_1$
  (becomes qDES for phase-dependent damping, $D_1 = D_1(\theta_1)$)
- frequency regulation and synchronization meet IEEE standards for realistic disturbance values[1]

[1] Ajala, Dominguez-Garcia, L, Robust leader-follower synchronization of electric power generators, SCL, 2021

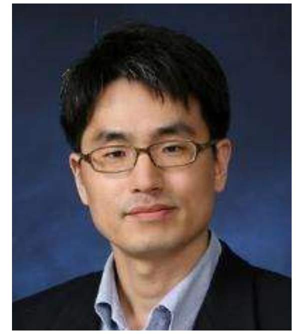
ACKNOWLEDGEMENTS

Entropy:

Sayan Mitra

Guosong Yang

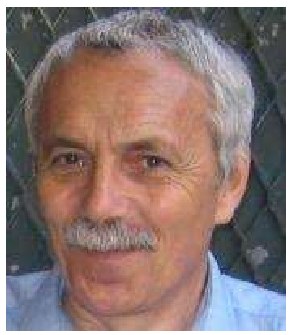
Observers:



Hyungbo Shim

Synchronization:

Boris Andrievsky



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